

QUANTUM UNIQUE ERGODICITY OF DEGENERATE EISENSTEIN SERIES ON $GL(n)$

LIYANG ZHANG

ABSTRACT. We prove quantum unique ergodicity for a subspace of the continuous spectrum spanned by the degenerate Eisenstein Series on $GL(n)$.

CONTENTS

1. Introduction	1
1.1. Introduction	1
1.2. Strategy for proof of Theorem 1.1	3
2. Automorphic Forms on $GL(n)$	4
2.1. Automorphic Functions, Automorphic Forms, and Fourier Expansion	4
2.2. Parabolic Subgroups and Eisenstein Series	6
2.3. Incomplete Eisenstein Series and Spectral Decomposition	9
3. Fourier Expansion of Degenerate Eisenstein Series	11
3.1. Fourier Expansion of Degenerate Eisenstein Series	11
3.2. Constant Term Computation	15
4. Cuspidal and Non-Minimal Eisenstein Contribution	19
4.1. Some Basic Lemmata	19
4.2. Partition of the Form $n = n_1 + \cdots + n_{r-1} + 1$	20
4.3. A $GL(2)$ Calculation	26
4.4. Cuspidal Contribution	27
4.5. Partition of type $n = n_1 + \cdots + n_r, n_r \geq 2$.	29
5. Main Term From Minimal Parabolic Contribution	32
References	42

1. INTRODUCTION

1.1. Introduction. In the classical setting, the evolution of a dynamical system (X, μ, T) can be described by the geodesic flow:

$$g_t : T^*X \rightarrow T^*X$$

where $g_t \mathbf{x}_0 = \mathbf{x}_t$ is given by the Hamiltonian. We say the system is ergodic if for every $f \in L^2_\mu$ and for almost every starting position \mathbf{x}_0 the time average becomes the spatial average:

$$\lim_{S \rightarrow \infty} \frac{1}{S} \int_0^S f(g_t(\mathbf{x}_0)) dt = \frac{1}{\mu(X)} \int_X f(\mathbf{x}) d\mathbf{x}.$$

A classic example of an ergodic dynamical system is the Bunimovich Stadium [Bu][BS].

Key words and phrases. Quantum Unique Ergodicity, $GL(n)$ Eisenstein Series, Incomplete Eisenstein Series,

The author was supported by the following grants from Alex Kontorovich: NSF CAREER grant DMS-1254788 and DMS-1455705.

In the quantum setting, the evolution of a system is governed by the Schrödinger equation:

$$-\frac{\hbar^2}{2m}\Delta\psi_n = \lambda_n\psi_n$$

where Δ is the Laplacian. A system is quantum uniquely ergodic if in the semiclassical limit ($\hbar \rightarrow 0$), each individual eigenfunction $|\psi_n|^2$ become equi-distributed.

Let (M, μ) be a Riemannian manifold with laplacian Δ and let ψ_n a set of orthonormal eigenfunctions of Δ with corresponding eigenvalues $0 \leq \lambda_1 \leq \lambda_2 \leq \dots$. Schnirelman [Sh], Colin de Verdière [Co] and Zelditch [Ze1] proved quantum ergodicity: eigenfunctions of Δ become equi-distributed with respect to the volume measure in the high energy limit along a subsequence n_k of density one. Zelditch [Ze2] extended this result to the modular surface, which is not compact. Hejhal-Rackner [HR], Rudnick-Sarnak [RS] conjectured that there are no exceptional subsequences, that is, there is quantum unique ergodicity. The arithmetic version of the conjecture was famously proved by Lindenstrauss [Li] and Soundararajan [S]. For higher rank, Silberman and Venkatesh formulated and proved some cases of quantum unique ergodicity for compact locally symmetric spaces in [SV1] and [SV2].

For non-compact quotients, the Laplacian Δ has continuous spectrum and one can formulate QUE for such. For example, on the modular surface the continuous spectrum is spanned by the Eisenstein series

$$E(z, s) = \sum_{\gamma \in \Gamma_\infty \backslash SL(2, \mathbb{Z})} \Im(\gamma z)^s.$$

Arithmetic quantum unique ergodicity of Eisenstein series on $GL(2)$ was first formulated and proved by Luo and Sarnak in [LS]. The unitary Eisenstein series $E(z, \frac{1}{2} + it)$ is an eigenfunction of the hyperbolic Laplacian with eigenvalue $\frac{1}{4} + t^2$. Define $d\nu_t = |E(z, \frac{1}{2} + it)|^2 \frac{dx dy}{y^2}$. Then for compact Jordan measurable sets A, B in $SL(2, \mathbb{Z}) \backslash SL(2, \mathbb{R}) / SO(2, \mathbb{R})$, Luo and Sarnak showed

$$\lim_{t \rightarrow \infty} \frac{\nu_t(A)}{\nu_t(B)} = \frac{\text{Vol}(A)}{\text{Vol}(B)}$$

which represents arithmetic quantum unique ergodicity for the continuous spectrum. More precisely, they showed

$$\nu_t(A) = \frac{12}{\pi} \text{Vol}(A) \log t + O(1)$$

as $t \rightarrow \infty$. Note that ergodic methods typically do not give rates. We also remark that the constant $\frac{12}{\pi}$ is different from that in [LS] because of different normalization.

In this paper, we study the analog of Luo and Sarnak's result on a subspace of the $GL(n)$ continuous spectrum spanned by the degenerate Eisenstein series induced from the maximal parabolic subgroup with the constant function $E_{n-1,1}(z, s, 1)$. Eisenstein series are eigenfunctions of Casimir operators and on $GL(n)$, the analog of the hyperbolic Laplacian is $\Delta_{2,n}$ (see section 2) where

$$\left(\Delta_{2,n} + \left(\frac{n^2 - n}{8} + \frac{n^2 - n}{2} t^2 \right) \right) E_{n-1,1} \left(z, \frac{1}{2} + it, 1 \right) = 0.$$

Define $\mu_{n,t} = |E_{n-1,1}(z, \frac{1}{2} + it, 1)|^2 d^*z$ where d^*z is the Haar measure. We will show the following:

Theorem 1.1. *Let A, B be compact Jordan measurable subsets of $SL(n, \mathbb{Z}) \backslash SL(n, \mathbb{R}) / SO(n, \mathbb{R})$. Then for $n \geq 2$,*

$$\lim_{t \rightarrow \infty} \frac{\mu_{n,t}(A)}{\mu_{n,t}(B)} = \frac{\text{Vol}(A)}{\text{Vol}(B)}.$$

More precisely,

$$\mu_{n,t}(A) = \frac{2}{\xi(n)} \text{Vol}(A) \log t + O_A(1)$$

as $t \rightarrow \infty$.

Remark 1.1. This theorem will serve as a stepping stone towards proving quantum unique ergodicity of higher rank non-degenerate Eisenstein series. QUE of higher rank non-degenerate Eisenstein series is a much more difficult problem as it is related to the shifted convolution problem involving generalized divisor functions and Fourier coefficients of higher rank Maass cusp forms.

1.2. Strategy for proof of Theorem 1.1. We use the spectral decomposition of $\mathcal{L}^2(SL_n(\mathbb{Z}) \backslash X_n)$ to divide the proof of Theorem 1.1 into evaluating the following three type of integrals (see Theorem 2.2):

$$\begin{aligned} & \int_{SL_n(\mathbb{Z}) \backslash X_n} \phi(z) \left| E_{n-1,1} \left(z, \frac{1}{2} + it, 1 \right) \right|^2 d^*z, \\ & \int_{SL_n(\mathbb{Z}) \backslash X_n} E_{1,\dots,1}(z, \eta) \left| E_{n-1,1} \left(z, \frac{1}{2} + it, 1 \right) \right|^2 d^*z, \\ & \int_{SL_n(\mathbb{Z}) \backslash X_n} E_{n_1,\dots,n_r}(z, \psi, u_1, \dots, u_r) \left| E_{n-1,1} \left(z, \frac{1}{2} + it, 1 \right) \right|^2 d^*z. \end{aligned}$$

Here $\phi(z)$ is a $GL(n)$ cusp form, $E_{1,\dots,1}(z, \eta)$ is an incomplete Eisenstein series associated to the minimal parabolic subgroup, and $E_{n_1,\dots,n_r}(z, \psi, u_1, \dots, u_r)$ is an incomplete Eisenstein series associated to the (n_1, \dots, n_r) parabolic subgroup induced from cusp forms u_j on $GL(n_j)$. These integrals will be evaluated in sections 4 and 5.

Acknowledgement. I would like to thank Alex Kontorovich for suggesting the problem, many enlightening discussions, and careful readings of the various versions of this paper. I also would like to thank Valentin Blomer for helpful comments.

2. AUTOMORPHIC FORMS ON $GL(n)$

Here we give a brief introduction to automorphic forms on $GL(n)$, make some preliminary computations and set the notation for the rest of the paper.

2.1. Automorphic Functions, Automorphic Forms, and Fourier Expansion. We are working over the generalized upper half space $X_n := GL_n(\mathbb{R}) / (O_n(\mathbb{R}) \cdot \mathbb{R}^*)$. By the Iwasawa decomposition (see section 1.2 of [Go]), X_n consists of matrices of the form $z = x \cdot y$ with

$$(2.1) \quad x = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \cdots & x_{1,n} \\ & 1 & x_{2,3} & \cdots & x_{2,n} \\ & & \ddots & & \vdots \\ & & & 1 & x_{n-1,n} \\ & & & & 1 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 y_2 \cdots y_{n-1} & & & & \\ & y_1 y_2 \cdots y_{n-2} & & & \\ & & \ddots & & \\ & & & y_1 & \\ & & & & 1 \end{pmatrix}.$$

where $x_{i,j} \in \mathbb{R}$ and $y_k > 0$. We say a function f is automorphic if

$$f(\gamma z) = f(z) \quad \forall \gamma \in SL_n(\mathbb{Z}), \forall z \in X_n.$$

The space X_n is equipped with a left $GL_n(\mathbb{R})$ -invariant measure d^*z on X_n given explicitly by

$$d^*z = c_n d^*x d^*y$$

where

$$c_n = n^{-1} \prod_{\ell=2}^n \xi(\ell)^{-1}, \quad d^*x = \prod_{1 \leq i < j \leq n} dx_{i,j}, \quad \prod_{k=1}^{n-1} d_k^{-k(n-k)} \frac{dy_k}{y_k}.$$

Throughout this paper, we define the completed zeta-function $\xi(s)$ to be $\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)$. The normalization for this measure is chosen so that

$$\int_{SL_n(\mathbb{Z}) \backslash X_n} 1 d^*z = 1.$$

Our interest lies in the Hilbert space $\mathfrak{L}(SL_n(\mathbb{Z}) \backslash X_n)$ where the inner product is defined by

$$\langle f, g \rangle = \int_{SL_n(\mathbb{Z}) \backslash X_n} f(z) \overline{g}(z) d^*z.$$

For a smooth function f in $\mathfrak{L}(SL_n(\mathbb{Z}) \backslash X_n)$, standard Fourier theory with the automorphy of f give rise to a Fourier expansion of f .

Theorem 2.1. (See section 5.3 of [Go].) Let $P_{n-1,1}(\mathbb{Z}) = \left\{ \begin{pmatrix} & * & & & \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \in SL_n(\mathbb{Z}) \right\}$. For $f \in \mathfrak{L}(SL_n(\mathbb{Z}) \backslash X_n)$ a smooth function, we have for all $z \in SL_n(\mathbb{Z}) \backslash X_n$

$$(2.2) \quad f(z) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \sum'_{\gamma_2 \in P_{1,1} \backslash SL_2(\mathbb{Z})} \cdots \sum_{m_{n-1}=0}^{\infty} \sum'_{\gamma_{n-1} \in P_{n-2,1} \backslash SL_{n-1}(\mathbb{Z})} \hat{f}_{(m_1, \dots, m_{n-1})}(z) \Big|_{\gamma_2 \cdots \gamma_{n-1}}.$$

where the slash operator is defined by

$$g(z)|_{\gamma_2 \cdots \gamma_{n-1}} = g \left(\begin{pmatrix} \gamma_2 & \\ & I_{n-2} \end{pmatrix} \cdots \begin{pmatrix} \gamma_{n-1} & \\ & 1 \end{pmatrix} \cdot z \right),$$

and for each $2 \leq h \leq n-1$, the primed summation over each $P_{h-1,1} \backslash SL_h(\mathbb{Z})$ is summed only if $m_h \neq 0$. Let $U_n(\mathbb{Z})$ ($U_n(\mathbb{R})$) denote the group of $n \times n$ upper triangular matrices with integer (real) entries and 1's on the diagonal

$$\hat{f}_{(m_1, \dots, m_{n-1})}(z) = \int_0^1 \cdots \int_0^1 \phi(u \cdot z) e(-m_1 u_{1,2} - m_2 u_{2,3} - \cdots - m_{n-1} u_{n-1,n}) \prod_{1 \leq i < j \leq n} du_{i,j}$$

and

$$u = \begin{pmatrix} 1 & u_{1,2} & u_{1,3} & \cdots & u_{1,n} \\ & 1 & u_{2,3} & \cdots & u_{2,n} \\ & & \ddots & & \vdots \\ & & & 1 & u_{n-1,n} \\ & & & & 1 \end{pmatrix} \in U_n(\mathbb{R}).$$

Our goal is to describe a spectral decomposition of the space $\mathfrak{L}(SL_n(\mathbb{Z}) \backslash X_n)$. It is natural to turn to the theory of differential operators. Let \mathfrak{D}_n be the center of the universal enveloping algebra of the Lie algebra $\mathfrak{gl}_n(\mathbb{R})$. Let $E_{i,j} \in \mathfrak{gl}_n(\mathbb{R})$ denote the matrix with a 1 at the (i,j) entry and zeros elsewhere. We define the differential operator $D_{i,j}$ by

$$(D_{i,j}f)(g) = \left. \frac{\partial}{\partial t} f(g \cdot \exp(tE_{i,j})) \right|_{t=0}$$

for a smooth function $f : GL_n(\mathbb{R}) \mapsto \mathbb{C}$.

Proposition 2.1. (See section 2.3 of [Go].) For $n \geq 2$ and $2 \leq m \leq n$, the differential operators (Casimir operators)

$$\Delta_{m,n} := \sum_{i_1=1}^n \sum_{i_2=1}^n \cdots \sum_{i_m=1}^n D_{i_1, i_2} \circ D_{i_2, i_3} \circ \cdots \circ D_{i_m, i_1}$$

generate \mathfrak{D}_n as a polynomial algebra of rank $n-1$.

For $n \geq 2$ and $\nu = (\nu_1, \dots, \nu_{n-1}) \in \mathbb{C}^{n-1}$, the I -function defined by

$$I_\nu(z) = \prod_{i=1}^{n-1} \prod_{j=1}^{n-1} y_i^{b_{i,j} \nu_j},$$

where

$$b_{i,j} = \begin{cases} ij & \text{if } i+j \leq n, \\ (n-i)(n-j) & \text{if } i+j \geq n \end{cases}$$

is an eigenfunction of all differential operators in \mathfrak{D}_n . Let $\lambda_{m,n}$ be the corresponding eigenvalues, i.e.

$$\Delta_{m,n} I_\nu(z) = \lambda_{m,n} I_\nu(z).$$

Definition 2.1. An automorphic form $\phi(z) \in \mathcal{L}^2(SL_n(\mathbb{Z}) \backslash X_n)$ of spectral type $\nu \in \mathbb{C}^{n-1}$ is a smooth function satisfying:

- (1) $\phi(\gamma z) = \phi(z)$, $\forall \gamma \in SL_n(\mathbb{Z}), \forall z \in X_n$,
- (2) $\Delta_{m,n} \phi(z) = \lambda_{m,n} \phi(z)$;

if $\phi(z)$ also satisfies

$$\int_{(SL_n(\mathbb{Z}) \cap U) \backslash U} \phi(u \cdot z) du = 0,$$

for all matrices of the form

$$U = \left\{ \begin{pmatrix} I_{r_1} & & & \\ & I_{r_2} & & * \\ & & \ddots & \\ & & & I_{r_m} \end{pmatrix} \right\} \subset SL_n(\mathbb{R})$$

with $r_1 + \cdots + r_m = n$, then $\phi(z)$ is a Maass form.

For an automorphic form $\phi(z)$ of type ν , the Fourier expansion (2.2) can be made more explicit by the theory of Whittaker functions. Multiplicity one theorem implies that each $\hat{\phi}_{(m_1, \dots, m_{n-1})}(z)$ can be expressed in terms of Whittaker functions:

$$W_{(m_1, \dots, m_{n-1})}^\nu(z, w) := \int_{U_n(\mathbb{R})} I_\nu(w \cdot u \cdot z) e(-m_1 u_{1,2} - \cdots - m_{n-1} u_{n-1,n}) \prod_{1 \leq i < j \leq n} du_{i,j},$$

where $w \in W_n$ is an element of the Weyl group. If $\phi(z)$ is a Maass cusp form, then (2.2) can be further simplified to

$$\phi(z) = \sum_{\gamma \in U_{n-1}(\mathbb{Z}) \backslash SL_{n-1}(\mathbb{Z})} \sum_{m_1=1}^{\infty} \cdots \sum_{m_{n-2}=1}^{\infty} \sum_{m_{n-1} \neq 0} a_{(m_1, \dots, m_{n-1})} W_{(m_1, \dots, m_{n-1})}^\nu(z, w_l) \Big|_{\gamma},$$

where w_l is the long Weyl element $\begin{pmatrix} & & & 1 \\ & & 1 & \\ & \ddots & & \\ 1 & & & \end{pmatrix}.$

2.2. Parabolic Subgroups and Eisenstein Series. Here we give a summary of Langlands' theory of Eisenstein series associated to Maass forms.

Definition 2.2. The standard parabolic subgroup $P_{n_1, \dots, n_r}(\mathbb{R}) \subset GL_n(\mathbb{R})$ associated to the partition $n = n_1 + n_2 + \cdots + n_r$ is defined to be the group of matrices of the form

$$\begin{pmatrix} \mathfrak{m}_{n_1} & * & \cdots & * \\ 0 & \mathfrak{m}_{n_2} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathfrak{m}_{n_r} \end{pmatrix},$$

where $\mathfrak{m}_{n_i} \in GL_{n_i}(\mathbb{R})$ for $1 \leq i \leq r$. We also define $P_{n_1, \dots, n_r}(\mathbb{Z}) = P_{n_1, \dots, n_r}(\mathbb{R}) \cap SL_n(\mathbb{Z})$. Two parabolic subgroups $P_{n_1, \dots, n_r}(\mathbb{R}), P_{n'_1, \dots, n'_r}(\mathbb{R})$ of $GL_n(\mathbb{R})$ are said to be associate if the set $\{n_1, \dots, n_r\}$ is a permutation of the set $\{n'_1, \dots, n'_r\}$.

A parabolic subgroup $P_{n_1, \dots, n_r}(\mathbb{R})$ can be decomposed into

$$P_{n_1, \dots, n_r}(\mathbb{R}) = N_{n_1, \dots, n_r}(\mathbb{R}) M_{n_1, \dots, n_r}(\mathbb{R}),$$

where

$$N_{n_1, \dots, n_r}(\mathbb{R}) = \left\{ \begin{pmatrix} I_{n_1} & * & \cdots & * \\ 0 & I_{n_2} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_{n_r} \end{pmatrix} \in GL_n(\mathbb{R}), I_k \text{ is an } k \times k \text{ identity matrix} \right\}$$

is the unipotent radical and

$$M_{n_1, \dots, n_r}(\mathbb{R}) = \left\{ \begin{pmatrix} \mathfrak{m}_{n_1} & 0 & \cdots & 0 \\ 0 & \mathfrak{m}_{n_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathfrak{m}_{n_r} \end{pmatrix}, \mathfrak{m}_k \in GL_{n_k}(\mathbb{R}) \right\}$$

is the Levi component. This is the Langlands decomposition of parabolic subgroups. We define $N_{n_1, \dots, n_r}(\mathbb{Z}) = N_{n_1, \dots, n_r}(\mathbb{R}) \cap SL_n(\mathbb{Z})$, $M_{n_1, \dots, n_r}(\mathbb{Z}) = M_{n_1, \dots, n_r}(\mathbb{R}) \cap SL_n(\mathbb{Z})$. For $g \in P_{n_1, \dots, n_r}(\mathbb{R})$, Langlands decomposition naturally gives rise to the projection maps $\mathfrak{m}_{n_i} : P_{n_1, \dots, n_r}(\mathbb{R}) \mapsto GL_{n_i}(\mathbb{R})$ by

$$g = \begin{pmatrix} I_{n_1} & * & \cdots & * \\ 0 & I_{n_2} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_{n_r} \end{pmatrix} \cdot \begin{pmatrix} \mathfrak{m}_{n_1}(g) & 0 & \cdots & 0 \\ 0 & \mathfrak{m}_{n_2}(g) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathfrak{m}_{n_r}(g) \end{pmatrix}.$$

To describe Eisenstein series associated to Maass forms we also need to define I -functions associated to parabolic subgroups.

Definition 2.3. Let $s = (s_1, \dots, s_r) \in \mathbb{C}^r$ satisfying $\sum_{i=1}^r n_i s_i = 0$ and for $z \in X_n$ in Iwasawa form (2.1) we define

$$I_s(z, P_{n_1, \dots, n_r}) = \left(\prod_{j_1=n-n_1+1}^n Y_{j_1} \right)^{s_1} \cdot \left(\prod_{j_2=n-n_1-n_2+1}^{n-n_1} Y_{j_2} \right)^{s_2} \cdots \left(\prod_{j_r=1}^{n_r} Y_{j_r} \right)^{s_r},$$

where Y_1, Y_2, \dots, Y_n are defined by

$$\begin{pmatrix} Y_n & & & \\ & Y_{n-1} & & \\ & & \ddots & \\ & & & Y_1 \end{pmatrix} = \begin{pmatrix} y_1 y_2 \cdots y_{n-1} & & & \\ & y_1 y_2 \cdots y_{n-2} & & \\ & & \ddots & \\ & & & y_1 \\ & & & & 1 \end{pmatrix}.$$

Definition 2.4. Let $P_{n_1, \dots, n_r}(\mathbb{R})$ be a parabolic subgroup of $GL_n(\mathbb{R})$ with projection maps \mathfrak{m}_{n_i} defined by Langlands decomposition. Let ϕ_i be Maass forms on $GL_{n_i}(\mathbb{R})$ for $1 \leq i \leq r$, and let $s = (s_1, \dots, s_r) \in \mathbb{C}^r$ satisfy

$$\sum_{i=1}^r n_i s_i = 0.$$

We define the Eisenstein series $E_{(n_1, \dots, n_r)}(z, s, \phi_1, \dots, \phi_r)$ by

$$E_{(n_1, \dots, n_r)}(z, s, \phi_1, \dots, \phi_r) = \sum_{\gamma \in P_{n_1, \dots, n_r}(\mathbb{Z}) \backslash SL_n(\mathbb{Z})} \prod_{i=1}^r \phi_i(\mathfrak{m}_i(\gamma z)) I_s(\gamma z, P_{n_1, \dots, n_r}).$$

Because of the relation $\sum_{i=1}^r n_i s_i = 0$, we may eliminate s_r . So $E_{n_1, \dots, n_r}(z, s, \phi_1, \dots, \phi_r)$ is really a function of z and $s = (s_1, \dots, s_{r-1}) \in \mathbb{C}^{r-1}$ and we will use this convention for the remaining part of this paper.

We want to especially point out that if we use the partition $n = 1 + \dots + 1$, we get the minimal parabolic Eisenstein series

$$E_{(1,\dots,1)}(z, s) = \sum_{\gamma \in P_{1,\dots,1}(\mathbb{Z}) \backslash SL_n(\mathbb{Z})} I_s(\gamma z).$$

For the maximal parabolic $P_{n-1,1}$, we can define the totally degenerate Eisenstein series

$$E_{(n-1,1)}(z, s, 1) = \sum_{\gamma \in P_{n-1,1}(\mathbb{Z}) \backslash SL_n(\mathbb{Z})} I_s(\gamma z, P_{n-1,1}).$$

This is the main object of study in this paper.

Langlands' theory of constant terms along arbitrary parabolic subgroups play a central role in understanding these Eisenstein series. We will use this theory to understand the Fourier expansion of some Eisenstein series.

Proposition 2.2. *Let $n = n_1 + \dots + n_r$ be a partition in descending order with $n_r \geq 2$, and let $E_{n_1,\dots,n_r}(z, s, \phi_1, \dots, \phi_r)$ be an Eisenstein series. Then its constant terms along $P_{(n-1,1)}$, $P_{(n-3,1,2)}$, $P_{(1,\dots,1)}$ are zero:*

$$\begin{aligned} \int_{N_{n-1,1}(\mathbb{Z}) \backslash N_{n-1,1}(\mathbb{R})} E_{(n_1,\dots,n_r)}(z, s, \phi_1, \dots, \phi_r) &= 0, \\ \int_{N_{n-3,1,2}(\mathbb{Z}) \backslash N_{n-3,1,2}(\mathbb{R})} E_{(n_1,\dots,n_r)}(z, s, \phi_1, \dots, \phi_r) &= 0, \\ \int_{N_{1,\dots,1}(\mathbb{Z}) \backslash N_{1,\dots,1}(\mathbb{R})} E_{(n_1,\dots,n_r)}(z, s, \phi_1, \dots, \phi_r) &= 0. \end{aligned}$$

Proof. This is a direct result of the proposition in II.1.7 of [MW]. Since $n_i \geq 2$ for all $1 \leq i \leq r$, $wM_{n_1,\dots,n_r}(\mathbb{R})w^{-1}$ cannot be contained in either $M_{(n-1,1)}(\mathbb{R})$, $M_{(n-3,1,2)}(\mathbb{R})$ or $M_{(1,\dots,1)}(\mathbb{R})$ for any Weyl element w . \square

Corollary 2.1. *Let $n = n_1 + \dots + n_r$ be a partition in descending order with $n_r \geq 2$, then in the Fourier expansion of $E_{(n_1,\dots,n_r)}(z, s, \phi_1, \dots, \phi_r)$, the Fourier coefficients $a_{(m_1,\dots,m_{n-2},0)}$, $a_{(m_1,\dots,m_{n-4},0,0,m_{n-1})}$ and $a_{(0,\dots,0)}$ are all zero.*

Proof. Let $f(z)E_{(n_1, \dots, n_r)}(z, s, \phi_1, \dots, \phi_r)$. We have

$$\begin{aligned}
\hat{f}_{(m_1, \dots, m_{n-2}, 0)}(z) &= \int_0^1 \cdots \int_0^1 f(u \cdot z) e(-m_1 u_{1,2} - m_2 u_{2,3} - \cdots - m_{n-2} u_{n-2, n-1}) \prod_{1 \leq i < j \leq n} du_{i,j} \\
&= \int_0^1 \cdots \int_0^1 f \left(\begin{pmatrix} 1 & 0 & 0 & \cdots & u_{1,n} \\ & 1 & 0 & \cdots & u_{2,n} \\ & & \ddots & \cdots & \vdots \\ & & & 1 & u_{n-1,n} \\ & & & & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & u_{1,2} & \cdots & u_{1,n-1} & 0 \\ & 1 & \cdots & u_{2,n-1} & 0 \\ & & \ddots & \vdots & \vdots \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix} \cdot z \right) \prod_{1 \leq i \leq n-1} du_{i,n} \\
&\quad \times e(-m_1 u_{1,2} - m_2 u_{2,3} - \cdots - m_{n-2} u_{n-2, n-1}) \prod_{1 \leq i < j \leq n-1} du_{i,j} \\
&= \int_0^1 \cdots \int_0^1 \int_{N_{n-1,1}(\mathbb{Z}) \backslash N_{n-1,1}(\mathbb{R})} f \left(u' \cdot \begin{pmatrix} 1 & u_{1,2} & \cdots & u_{1,n-1} & 0 \\ & 1 & \cdots & u_{2,n-1} & 0 \\ & & \ddots & \vdots & \vdots \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix} \cdot z \right) \prod_{1 \leq i \leq n-1} du'_{i,n} \\
&\quad \times e(-m_1 u_{1,2} - m_2 u_{2,3} - \cdots - m_{n-2} u_{n-2, n-1}) \prod_{1 \leq i < j \leq n-1} du_{i,j} \\
&= 0,
\end{aligned}$$

by Proposition 2.2. For $\hat{f}_{(m_1, \dots, m_{n-4}, 0, 0, m_{n-1})}$, the proof is similar as we have

$$\begin{aligned}
u &= \begin{pmatrix} 1 & 0 & \cdots & 0 & u_{1,n-2} & u_{1,n-1} & u_{1,n} - u_{1,n-1}u_{n-1,n} \\ & 1 & \ddots & \vdots & u_{2,n-2} & u_{2,n-1} & u_{2,n} - u_{2,n-1}u_{n-1,n} \\ & & \ddots & 0 & \vdots & \vdots & \vdots \\ & & & 1 & u_{n-3,n-2} & u_{n-3,n-1} & u_{n-3,n} - u_{n-3,n-1}u_{n-1,n} \\ & & & & 1 & u_{n-2,n-1} & u_{n-2,n} - u_{n-2,n-1}u_{n-1,n} \\ & & & & & 1 & 0 \\ & & & & & & 1 \end{pmatrix} \\
&\quad \times \begin{pmatrix} 1 & u_{1,2} & \cdots & u_{1,n-3} & 0 & 0 & 0 \\ & 1 & \cdots & u_{2,n-3} & 0 & 0 & 0 \\ & & \ddots & \vdots & \vdots & \vdots & \vdots \\ & & & 1 & 0 & 0 & 0 \\ & & & & 1 & 0 & 0 \\ & & & & & 1 & u_{n-1,n} \\ & & & & & & 1 \end{pmatrix}.
\end{aligned}$$

A simple change of variable $u_{i,n} - u_{i,n-1}u_{n-1,n} \mapsto u_{i,n}$ with the fact that f is automorphic give the desired result again by Proposition 2.2. Finally $a_{(0, \dots, 0)} = 0$ is a direct result of the last part of Proposition 2.2. \square

2.3. Incomplete Eisenstein Series and Spectral Decomposition. Eisenstein series induced from Maass forms play a central role in the spectral decomposition of the space $\mathcal{L}(SL_n(\mathbb{Z}) \backslash X_n)$. But these

Eisenstein series fail to be in $\mathfrak{L}(SL_n(\mathbb{Z}) \backslash X_n)$. So we need the theory of incomplete Eisenstein series. (They are referred as pseudo-Eisenstein in [MW]). Incomplete Eisenstein series were used in the study of quantum ergodicity of $GL(2)$ Eisenstein series in [LS]. The analogous theory of incomplete Eisenstein series in more general settings can be found in [L, MW, Ve]. Here we give a brief summary for this theory on $GL(n)$ explicitly. First we clarify the definition of Mellin transform.

Definition 2.5. For $\eta \in C_0^\infty((\mathbb{R}^+)^n)$, we define the n -dimensional Mellin transform of η to be

$$\tilde{\eta}(s_1, \dots, s_n) = \int_0^\infty \cdots \int_0^\infty \eta(y_1, \dots, y_n) y_1^{-s_1} \cdots y_n^{-s_n} \frac{dy_1 \cdots dy_n}{y_1 \cdots y_n}.$$

Definition 2.6. For $\eta \in C_0^\infty((\mathbb{R}^+)^{r-1})$, we define the incomplete Eisenstein series $E_{(n_1, \dots, n_r)}(z, \eta, \phi_1, \dots, \phi_r)$ associated to $E_{(n_1, \dots, n_r)}(z, s, \phi_1, \dots, \phi_r)$ by

$$E_{(n_1, \dots, n_r)}(z, \eta, \phi_1, \dots, \phi_r) = \frac{1}{(2\pi i)^{r-1}} \int_{(2)} \cdots \int_{(2)} \tilde{\eta}(s_1, \dots, s_{r-1}) E_{(n_1, \dots, n_r)}(z, s, \phi_1, \dots, \phi_r) ds_1 \cdots ds_{r-1}.$$

Note that the convergence of these integrals in Definition 2.6 are guaranteed by the rapid decay of $\tilde{\eta}$ in imaginary parts of the arguments.

Let \mathcal{H}_{cusp} denote the space of $GL(n)$ cusp forms, $\mathcal{H}_{(n_1, \dots, n_r)}$ denote the space spanned by incomplete Eisenstein series $E_{(n_1, \dots, n_r)}(z, \eta, \phi_1, \dots, \phi_r)$. The following theorem is a summary of the results presented in Section II of [MW] applied to the group $GL(n)$.

Theorem 2.2. *We have the following spectral decomposition of automorphic forms on $GL(n)$:*

$$\mathfrak{L}(SL_n(\mathbb{Z}) \backslash X_n) = \mathcal{H}_{cusp} \oplus \bigoplus_{((n_1, \dots, n_r))} \mathcal{H}_{(n_1, \dots, n_r)}.$$

Note that $\mathcal{H}_{(n_1, \dots, n_r)} = \mathcal{H}_{(n'_1, \dots, n'_r)}$ if (n_1, \dots, n_r) and (n'_1, \dots, n'_r) are associate. So $((n_1, \dots, n_r))$ means that the decomposition is over representatives of associate partitions. For convenience, we always choose the partition in non-increasing order.

From this decomposition, we will prove our main theorem by estimating the following integrals as $t \rightarrow \infty$ as mentioned in the introduction:

$$\begin{aligned} & \int_{SL_n(\mathbb{Z}) \backslash X_n} \phi(z) \left| E_{n-1,1} \left(z, \frac{1}{2} + it, 1 \right) \right|^2 d^*z, \\ & \int_{SL_n(\mathbb{Z}) \backslash X_n} E_{1, \dots, 1}(z, \eta) \left| E_{n-1,1} \left(z, \frac{1}{2} + it, 1 \right) \right|^2 d^*z, \\ & \int_{SL_n(\mathbb{Z}) \backslash X_n} E_{n_1, \dots, n_r}(z, \psi, u_1, \dots, u_r) \left| E_{n-1,1} \left(z, \frac{1}{2} + it, 1 \right) \right|^2 d^*z. \end{aligned}$$

where $\phi(z)$ is a $GL(n)$ cusp form.

3. FOURIER EXPANSION OF DEGENERATE EISENSTEIN SERIES

3.1. Fourier Expansion of Degenerate Eisenstein Series. In this section, we present a complete calculation of the Fourier expansion of the degenerate Eisenstein series.

Theorem 3.1. *The degenerate Eisenstein series for $GL(n)$ has only degenerate terms in its Fourier-Whittaker expansion for $n \geq 3$. More precisely,*

$$E_{(n-1,1)}(z, s, 1) = \sum_{m_1 \in \mathbb{Z}} \hat{\phi}_{(m_1, 0, \dots, 0)}(z) + \sum_{i=2}^{n-1} \sum_{\gamma_i \in P_{(i-1,1)}(\mathbb{Z}) \backslash SL_i(\mathbb{Z})} \sum_{m_i=1}^{\infty} \hat{\phi}_{(0, \dots, 0, m_i, 0, \dots, 0)} \left(\begin{pmatrix} \gamma_i & \\ & I_{n-i} \end{pmatrix} z \right).$$

where

$$\hat{\phi}_{(m_1, \dots, m_{n-1})}(z, s) := \int_0^1 \cdots \int_0^1 E_{(n-1,1)}(u \cdot z, s, 1) e(-m_1 u_{1,2} - m_2 u_{2,3} - \cdots - m_{n-1} u_{n-1,n}) \prod_{1 \leq i < j \leq n} du_{i,j}$$

with $u \in U_n(\mathbb{R})$ and I_{n-i} is the identity matrix of dimension $n-i$.

Furthermore, the coefficients are calculated to be

$$\begin{aligned} \hat{\phi}_{(0, \dots, 0)}(z, s) &= \sum_{k=0}^{n-1} \frac{2\xi(ns - n + k + 1)}{\xi(ns)} \left(y_1 y_2^2 \cdots y_{n-k-1}^{n-k-1} \right)^{(1-s)} \left(y_{n-k}^k y_{n-(k-1)}^{k-1} \cdots y_{n-1} \right)^s; \\ \hat{\phi}_{(0, \dots, 0, m_k, 0, \dots, 0)}(z, s) &= e(m_k x_{k,k+1}) \frac{2}{\xi(ns)} |m_k|^{\frac{ns}{2} - \frac{n-k}{2}} \sigma_{-ns+n-k}(|m_k|) K_{\frac{ns}{2} - \frac{n-k}{2}}(2\pi |m_k| y_{n-k}) \\ &\quad \times \left(y_1 y_2^2 \cdots y_{n-k-1}^{n-k-1} \right)^{(1-s)} \left(y_{n-k}^k y_{n-(k-1)}^{k-1} \cdots y_{n-1} \right)^s y_{n-k}^{-\frac{ns}{2} + \frac{n-k}{2}}. \end{aligned}$$

We devote the reminding part of this section to proving the theorem. First rewrite the degenerate Eisenstein series in terms of Epstein Zeta function (see Section 10.7 of [Go] for a proof):

$$\zeta(ns) E_{(n-1,1)}(z, s, 1) = (y_1^{n-1} y_2^{n-2} \cdots y_{n-1})^s \sum_{(a_1, \dots, a_n) \in \mathbb{Z}^n \setminus \{0\}} (b_1^2 + \cdots + b_n^2)^{-ns/2}.$$

For

$$z = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \cdots & x_{1,n} \\ & 1 & x_{2,3} & \cdots & x_{2,n} \\ & & \ddots & & \vdots \\ & & & 1 & x_{n-1,n} \\ & & & & 1 \end{pmatrix} \begin{pmatrix} y_1 y_2 \cdots y_{n-1} & & & & \\ & y_1 y_2 \cdots y_{n-2} & & & \\ & & \ddots & & \\ & & & y_1 & \\ & & & & 1 \end{pmatrix},$$

the b_i 's are defined by

$$\begin{aligned} b_1 &= a_1 y_1 \cdots y_{n-1} \\ b_2 &= (a_1 x_{1,2} + a_2) y_1 \cdots y_{n-2} \\ &\vdots \\ b_n &= (a_1 x_{1,n} + a_2 x_{2,n} + \cdots + a_{n-1} x_{n-1,n} + a_n). \end{aligned}$$

Let

$$E_{(n-1,1)}^*(z, s) := \sum_{(a_1, \dots, a_n) \in \mathbb{Z}^n \setminus \{0\}} (b_1^2 + \cdots + b_n^2)^{-ns/2}.$$

First, we separate the terms with $a_1 = 0$:

$$\begin{aligned} E_{(n-1,1)}^*(z, s) &= \sum_{\substack{(a_2, \dots, a_n) \in \mathbb{Z}^{n-1} \setminus \{0\} \\ a_1=0}} (b_2^2 + \dots + b_n^2)^{-ns/2} + \sum_{a_1 \neq 0} \sum_{(a_2, \dots, a_n) \in \mathbb{Z}^{n-1}} (b_1^2 + \dots + b_n^2)^{-ns/2} \\ &= E_{(n-2,1)}^* \left(\pi_{n,n-1}(z), \frac{n}{n-1}s \right) + \mathcal{E}_n(z, s), \text{ say.} \end{aligned}$$

Here $\pi_{n,n-k} : X_n \rightarrow X_{n-k}$ is the projection map:

$$\begin{aligned} \pi : & \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \cdots & x_{1,n} \\ & 1 & x_{2,3} & \cdots & x_{2,n} \\ & & \ddots & & \vdots \\ & & & 1 & x_{n-1,n} \\ & & & & 1 \end{pmatrix} \begin{pmatrix} y_1 y_2 \cdots y_{n-1} & & & & \\ & y_1 y_2 \cdots y_{n-2} & & & \\ & & \ddots & & \\ & & & y_1 & \\ & & & & 1 \end{pmatrix} \\ \mapsto & \begin{pmatrix} 1 & x_{1+k,3} & x_{1+k,4} & \cdots & x_{1+k,n} \\ & 1 & x_{2+k,4} & \cdots & x_{2+k,n} \\ & & \ddots & & \vdots \\ & & & 1 & x_{n-1,n} \\ & & & & 1 \end{pmatrix} \begin{pmatrix} y_1 y_2 \cdots y_{n-1-k} & & & & \\ & y_1 y_2 \cdots y_{n-2-k} & & & \\ & & \ddots & & \\ & & & y_1 & \\ & & & & 1 \end{pmatrix}. \end{aligned}$$

By Poisson summation,

$$\mathcal{E}_n(z, s) = \sum_{a_1 \neq 0} \sum_{(c_2, \dots, c_n) \in \mathbb{Z}^{n-1}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{e(-c_2 a_2 - \cdots - c_n a_n)}{(b_1^2 + \dots + b_n^2)^{ns/2}} da_2 \cdots da_n.$$

Thus the Fourier-Whittaker coefficients of $E_{(n-1,1)}^*(z, s)$ are separated into two parts:

$$\begin{aligned} \frac{\zeta(ns)}{(y_1^{n-1} y_2^{n-2} \cdots y_{n-1})^s} \hat{\phi}_{(m_1, \dots, m_{n-1})}(z, s) &= \int_0^1 \cdots \int_0^1 \left(E_{(n-2,1)}^* \left(\pi_{n,n-1}(u \cdot z), \frac{n}{n-1}s \right) + \mathcal{E}_n(u \cdot z, s) \right) \\ &\quad \times e(-m_1 u_{1,2} - m_2 u_{2,3} - \cdots - m_{n-1} u_{n-1,n}) \prod_{1 \leq i < j \leq n} du_{i,j} \\ (3.1) \quad &= \hat{\phi}_{(m_1, \dots, m_{n-1})}^1(z, s) + \hat{\phi}_{(m_1, \dots, m_{n-1})}^2(z, s), \end{aligned}$$

say. We treat these two parts in the following two lemmata.

Lemma 3.1. *The first parts of the coefficients $\hat{\phi}_{(m_1, \dots, m_{n-1})}^1(z, s)$ have the following properties:*

- (1) $\hat{\phi}_{(m_1, \dots, m_{n-1})}^1(z, s) = 0$ if $m_1 \neq 0$;
- (2) $\hat{\phi}_{(0, m_2, \dots, m_{n-1})}^1(z, s) = \frac{\zeta(ns)}{(y_1^{n-2} y_2^{n-3} \cdots y_{n-2})^{\frac{n}{n-1}s}} \hat{\phi}_{(m_2, \dots, m_{n-1})} \left(\pi_{n,n-1}(z), \frac{n}{n-1}s \right).$

Proof. By direct matrix multiplication, we find that

$$(3.2) \quad \pi_{n,n-1}(u \cdot z) = \pi_{n,n-1}(u) \cdot \pi_{n,n-1}(z).$$

Thus the integral against the variable $u_{1,2}$ gives 0 unless $m_1 = 0$. This proves part (1).

For part (2), again by equation 3.2 the integrals against the variables $u_{1,j}$ for $2 \leq j \leq n$ have no effect. So

$$\begin{aligned}
& \int_0^1 \cdots \int_0^1 E_{(n-2,1)}^* \left(\pi_{n,n-1}(u \cdot z), \frac{n}{n-1}s \right) e(-m_1 u_{1,2} - m_2 u_{2,3} - \cdots - m_{n-1} u_{n-1,n}) \prod_{1 \leq i < j \leq n} du_{i,j} \\
&= \int_0^1 \cdots \int_0^1 E_{(n-2,1)}^* \left(\pi_{n,n-1}(u) \cdot \pi_{n,n-1}(z), \frac{n}{n-1}s \right) e(-m_2 u_{2,3} - \cdots - m_{n-1} u_{n-1,n}) \prod_{2 \leq i < j \leq n} du_{i,j} \\
&= \frac{\zeta(ns)}{(y_1^{n-2} y_2^{n-3} \cdots y_{n-2})^{\frac{n}{n-1}s}} \hat{\phi}_{(m_2, \dots, m_{n-1})} \left(\pi_{n,n-1}(z), \frac{n}{n-1}s \right).
\end{aligned}$$

□

Lemma 3.2. *The term $\hat{\phi}_{(m_1, \dots, m_{n-1})}^2(z, s)$ in equation 3.1 can be nonzero only if $m_2 = \cdots = m_{n-1} = 0$. For $m_1 \neq 0$*

$$\begin{aligned}
\hat{\phi}_{(m_1, 0, \dots, 0)}^2(z, s) &= e(m_1 x_{1,2}) \frac{2\pi^{\frac{ns}{2}}}{\Gamma\left(\frac{ns}{2}\right)} |m_1|^{\frac{ns}{2} - \frac{n-1}{2}} \sigma_{-ns+n-1}(|m_1|) (y_1^{n-3} \cdots y_{n-3})^{-1} (y_1 \cdots y_{n-2})^{-(ns-n+2)} \\
&\quad \times y_{n-1}^{-\frac{ns}{2} + \frac{n-1}{2}} K_{\frac{ns}{2} - \frac{n-1}{2}}(2\pi |m_1| y_{n-1});
\end{aligned}$$

for $m_1 = 0$

$$\begin{aligned}
& \hat{\phi}_{(0, \dots, 0)}^2(z, s) \\
&= \frac{2\pi^{\frac{n-1}{2}} \zeta(ns - n + 1) \Gamma\left(\frac{ns}{2} - \frac{n-1}{2}\right)}{\Gamma\left(\frac{ns}{2}\right)} (y_1^{n-2} \cdots y_{n-2})^{-1} (y_1 \cdots y_{n-1})^{-ns+n-1}.
\end{aligned}$$

Proof. Let x' denote the product of matrices $u \cdot x$. Define b'_i to be the same as b_i with x replaced by x' . We now make the change of variable $b'_n \mapsto a_n$:

$$\begin{aligned}
\hat{\phi}_{(m_1, \dots, m_{n-1})}^2(z, s) &= \int_0^1 \cdots \int_0^1 \sum_{a_1 \neq 0} \sum_{(c_2, \dots, c_n) \in \mathbb{Z}^{n-1}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \\
&\quad \frac{e(-c_2 a_2 - \cdots - c_n (a_n - a_1 x'_{1,n} - \cdots - a_{n-1} x'_{n-1,n}))}{(b_1'^2 + \cdots + a_n^2)^{ns/2}} da_2 \cdots da_n \\
&\quad \times e(-m_1 u_{1,2} - m_2 u_{2,3} - \cdots - m_{n-1} u_{n-1,n}) \prod_{1 \leq i < j \leq n} du_{i,j}.
\end{aligned}$$

We notice that the only entry with $u_{1,n}$ in the matrix x' is the $x'_{1,n}$ entry. As $a_1 \neq 0$, the integral against $u_{1,n}$ gives 0 unless $c_n = 0$. After this simplification we have

$$\begin{aligned}
\hat{\phi}_{(m_1, \dots, m_{n-1})}^2(z, s) &= \int_0^1 \cdots \int_0^1 \sum_{a_1 \neq 0} \sum_{(c_2, \dots, c_{n-1}) \in \mathbb{Z}^{n-2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{e(-c_2 a_2 - \cdots - c_{n-1} a_{n-1})}{(b_1'^2 + \cdots + a_n^2)^{ns/2}} da_2 \cdots da_n \\
&\quad \times e(-m_1 u_{1,2} - m_2 u_{2,3} - \cdots - m_{n-1} u_{n-1,n}) \frac{\prod_{1 \leq i < j \leq n} du_{i,j}}{du_{1,n}}
\end{aligned}$$

Now make the change of variable $\frac{b'_{n-1}}{y_1} \mapsto a_{n-1}$:

$$\begin{aligned} \hat{\phi}_{(m_1, \dots, m_{n-1})}^2(z, s) &= \int_0^1 \cdots \int_0^1 \sum_{a_1 \neq 0} \sum_{(c_2, \dots, c_{n-1}) \in \mathbb{Z}^{n-2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \\ &\quad \times \frac{e(-c_2 a_2 - \cdots - c_{n-1}(a_{n-1} - a_1 x'_{1,n-1} - \cdots - a_{n-2} x'_{n-2,n-1}))}{(b_1'^2 + \cdots + a_{n-1}^2 y_1^2 + a_n^2)^{ns/2}} da_2 \cdots da_n \\ &\quad \times e(-m_1 u_{1,2} - m_2 u_{2,3} - \cdots - m_{n-1} u_{n-1,n}) \frac{\prod_{1 \leq i < j \leq n} du_{i,j}}{du_{1,n}} \end{aligned}$$

The only term with $u_{1,n-1}$ is in the entry $x'_{1,n-1}$. Integrating against $u_{1,n-1}$ with the fact that $a_1 \neq 0$ shows $c_{n-1} = 0$. Continuing in this fashion, we get

$$\begin{aligned} (3.3) \quad \hat{\phi}_{(m_1, \dots, m_{n-1})}^2(z, s) &= \int_0^1 \cdots \int_0^1 \sum_{a_1 \neq 0} \sum_{c_2 \in \mathbb{Z}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{e(-c_2(a_2 - a_1 x_{1,2} - a_1 u_{1,2}))}{(a_1^2 y_1^2 \cdots y_{n-1}^2 + \cdots + a_n^2)^{ns/2}} da_2 \cdots da_n \\ &\quad \times e(-m_1 u_{1,2} - m_2 u_{2,3} - \cdots - m_{n-1} u_{n-1,n}) \frac{\prod_{1 \leq i < j \leq n} du_{i,j}}{du_{1,n} \cdots du_{1,3}}. \end{aligned}$$

Clearly, this is zero unless $m_2 = \cdots = m_{n-1} = 0$.

To continue, we will use the following well-known identities:

$$(3.4) \quad \int_{-\infty}^{\infty} (A^2 x^2 + C)^{-\nu} dx = \sqrt{\pi} \frac{\Gamma(\nu - \frac{1}{2})}{\Gamma(\nu)} |A|^{-1} C^{-\nu + \frac{1}{2}},$$

$$(3.5) \quad \int_{-\infty}^{\infty} (A^2 x^2 + C^2)^{-\nu} e(-Dx) dx = \frac{2\pi^\nu}{\Gamma(\nu)} |A|^{-\nu - \frac{1}{2}} |D|^{\nu - \frac{1}{2}} |C|^{\frac{1}{2} - \nu} K_{\nu - \frac{1}{2}} \left(\frac{2\pi |CD|}{A} \right),$$

where $A, C, D \in \mathbb{R}$. Continuing from 3.3, we have

$$\hat{\phi}_{(m_1, 0, \dots, 0)}^2(z, s) = \int_0^1 \sum_{a_1 \neq 0} \sum_{c_2 \in \mathbb{Z}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{e(-c_2(a_2 - a_1 x_{1,2} - a_1 u_{1,2}))}{(a_1^2 y_1^2 \cdots y_{n-1}^2 + \cdots + a_n^2)^{ns/2}} da_2 \cdots da_n e(-m_1 u_{1,2}) du_{1,2}.$$

For $m_1 = 0$,

$$\begin{aligned} \hat{\phi}_{(0, 0, \dots, 0)}^2(z, s) &= \int_0^1 \sum_{a_1 \neq 0} \sum_{c_2 \in \mathbb{Z}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{e(-c_2(a_2 - a_1 x_{1,2} - a_1 u_{1,2}))}{(a_1^2 y_1^2 \cdots y_{n-1}^2 + \cdots + a_n^2)^{ns/2}} da_2 \cdots da_n du_{1,2} \\ &= \sum_{a_1 \neq 0} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{1}{(a_1^2 y_1^2 \cdots y_{n-1}^2 + \cdots + a_n^2)^{ns/2}} da_2 \cdots da_n \\ &= \frac{2\pi^{\frac{n-1}{2}} \zeta(ns - n + 1) \Gamma(\frac{ns}{2} - \frac{n-1}{2})}{\Gamma(\frac{ns}{2})} (y_1^{n-2} \cdots y_{n-2})^{-1} (y_1 \cdots y_{n-1})^{-ns+n-1} \end{aligned}$$

after repeatedly applying 3.4. For $m_1 \neq 0$,

$$\begin{aligned} \hat{\phi}_{(m_1, 0, \dots, 0)}^2(z, s) &= \int_0^1 \sum_{a_1 \neq 0} \sum_{c_2 \in \mathbb{Z}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{e(-c_2(a_2 - a_1 x_{1,2} - a_1 u_{1,2}))}{(a_1^2 y_1^2 \cdots y_{n-1}^2 + \cdots + a_n^2)^{ns/2}} da_2 \cdots da_n e(-m_1 u_{1,2}) du_{1,2} \\ &= e(m_1 x_{1,2}) \sum_{\substack{a_1 \neq 0 \\ a_1 c_2 = m_1}} \sum_{c_2 \in \mathbb{Z}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{e(-c_2 a_2)}{(a_1^2 y_1^2 \cdots y_{n-1}^2 + \cdots + a_n^2)^{ns/2}} da_2 \cdots da_n \\ &= e(m_1 x_{1,2}) \frac{2\pi^{\frac{ns}{2}}}{\Gamma(\frac{ns}{2})} |m_1|^{\frac{ns}{2} - \frac{n-1}{2}} \sigma_{-ns+n-1}(|m_1|) (y_1^{n-3} \cdots y_{n-3})^{-1} (y_1 \cdots y_{n-2})^{-(ns-n+2)} \\ &\quad \times y_{n-1}^{-\frac{ns}{2} + \frac{n-1}{2}} K_{\frac{ns}{2} - \frac{n-1}{2}}(2\pi |m_1| y_{n-1}). \end{aligned}$$

□

Finally the calculation of the Fourier expansion is complete after a simple induction on n using equation 3.1.

3.2. Constant Term Computation. In this section, we utilize the Fourier expansion computation to investigate the following two constant terms:

$$\int_{(\mathbb{Z} \backslash \mathbb{R})^{n-1}} E_{(n-1,1)}(z, s, 1) \prod_{h=1}^{n-1} dx_{h,n},$$

$$\int_{(\mathbb{Z} \backslash \mathbb{R})^{n-1}} \left| E_{(n-1,1)} \left(z, \frac{1}{2} + it, 1 \right) \right|^2 \prod_{h=1}^{n-1} dx_{h,n}$$

to prepare for calculation in next section. First, we need a lemma.

Lemma 3.3. Let $\gamma_k = \begin{pmatrix} & & * & & \\ \gamma_{k,1} & \gamma_{k,2} & \cdots & \gamma_{k,k-1} & \gamma_{k,k} \end{pmatrix} \in SL_k(\mathbb{Z})$ and let

$$z' = \begin{pmatrix} 1 & x'_{1,2} & x'_{1,3} & \cdots & x'_{1,n} \\ & 1 & x'_{2,3} & \cdots & x'_{2,n} \\ & & \ddots & & \vdots \\ & & & 1 & x'_{n-1,n} \\ & & & & 1 \end{pmatrix} \cdot \begin{pmatrix} y'_1 y'_2 \cdots y'_{n-1} & & & & \\ & y_1 y_2 \cdots y_{n-2} & & & \\ & & \ddots & & \\ & & & y_1 & \\ & & & & 1 \end{pmatrix}$$

be the product $\begin{pmatrix} \gamma_k & \\ & I_{n-k} \end{pmatrix} \cdot z$ in Iwasawa form, then

(1)

$$y'_j = y_j \quad \text{for } 1 \leq j \leq n - k - 1;$$

(2)

$$y'^k_{n-k} y'^{k-1}_{n-(k-1)} \cdots y'_{n-1} = y^k_{n-k} y^{k-1}_{n-(k-1)} \cdots y_{n-1};$$

(3)

$$y'_{n-k} = y_{n-k} (b_1^2 + \cdots + b_k^2)^{\frac{1}{2}}$$

where

$$\begin{aligned} b_1 &= \gamma_{k,1} y_{n-k+1} \cdots y_{n-1} \\ b_2 &= (\gamma_{k,1} x_{1,2} + \gamma_{k,2}) y_{n-k+1} \cdots y_{n-2} \\ &\vdots \\ b_{k-1} &= (\gamma_{k,1} x_{1,k-1} + \gamma_{k,2} x_{2,k-1} + \cdots \gamma_{k,k-2} x_{k-2,k-1} + \gamma_{k,k-1}) y_{n-k+1} \\ b_k &= \gamma_{k,1} x_{1,k} + \gamma_{k,2} x_{2,k} + \cdots \gamma_{k,k-1} x_{k-1,k} + \gamma_{k,k}. \end{aligned}$$

(4)

$$x'_{k,k+1} = \sum_{i=1}^k \gamma_{k,i} x_{i,k+1}.$$

Proof. First we write

$$x = \begin{pmatrix} X_1 & V \\ & X_2 \end{pmatrix} \quad y = \begin{pmatrix} Y_1 y_1 \cdots y_{n-k-1} & \\ & Y_2 \end{pmatrix} = \begin{pmatrix} \tilde{Y}_1 y_1 \cdots y_{n-k} & \\ & Y_2 \end{pmatrix},$$

where $x \cdot y$ is the Iwasawa decomposition of z and X_1, Y_1 are of dimension $k \times k$, X_2, Y_2 are of dimension $(n-k) \times (n-k)$. Matrix multiplication gives

$$\begin{pmatrix} \gamma_k & \\ & I_{n-k} \end{pmatrix} \begin{pmatrix} X_1 & V \\ & X_2 \end{pmatrix} \begin{pmatrix} Y_1 y_1 \cdots y_{n-k-1} & \\ & Y_2 \end{pmatrix} = \begin{pmatrix} \gamma_k \cdot X_1 \cdot Y_1 & \gamma_k \cdot V \\ & X_2 \end{pmatrix} \begin{pmatrix} I_k y_1 \cdots y_{n-k-1} & \\ & Y_2 \end{pmatrix}.$$

By the Iwasawa decomposition, the matrix $\gamma_i \cdot X_1 \cdot Y_1$ can be written as $X'_1 \cdot Y'_1 \cdot K'_1$. So

$$\begin{pmatrix} \gamma_k & \\ & I_{n-k} \end{pmatrix} z = \begin{pmatrix} X'_1 & \gamma_k \cdot V \\ & X_2 \end{pmatrix} \begin{pmatrix} Y'_1 y_1 \cdots y_{n-k-1} & \\ & Y_2 \end{pmatrix} \begin{pmatrix} K' & \\ & 1 \end{pmatrix}.$$

We have put $\begin{pmatrix} \gamma_k & \\ & I_{n-k} \end{pmatrix} z$ in Iwasawa form. Note that $\det(Y_1) = \det(Y'_1) = y_{n-k}^k y_{n-(k-1)}^{k-1} \cdots y_{n-1}$ is invariant under the action of γ_k as $\gamma_k \in SL_k(\mathbb{Z})$ has determinant 1. At the same time, the variables $y_1, y_2, \dots, y_{n-k-1}$ are unchanged.

The proof of the final part of this lemma is presented on P.309 of [Go]. \square

Proposition 3.1.

$$\begin{aligned} & \int_{(\mathbb{Z} \setminus \mathbb{R})^{n-1}} E_{(n-1,1)}(z, s, 1) \prod_{h=1}^{n-1} dx_{h,n} \\ (3.6) \quad & = 2 (y_1^{n-1} \cdots y_{n-1})^s + \frac{\xi(ns-1)}{\xi(ns)} (y_1^{n-1} \cdots y_{n-1})^{\frac{1-s}{n-1}} E_{(n-2,1)} \left(\mathbf{m}_{n-1}(z), \frac{ns-1}{n-1}, 1 \right). \end{aligned}$$

Proof. By Theorem 3.1

$$\begin{aligned} & \int_{(\mathbb{Z} \setminus \mathbb{R})^{n-1}} E_{(n-1,1)}(z, s, 1) \prod_{h=1}^{n-1} dx_{h,n} \\ & = \int_{(\mathbb{Z} \setminus \mathbb{R})^{n-1}} \left(\sum_{m_1 \in \mathbb{Z}} \hat{\phi}_{(m_1, 0, \dots, 0)}(z) + \sum_{i=2}^{n-1} \sum_{\gamma_i \in P_{(i-1,1)}(\mathbb{Z}) \setminus SL_i(\mathbb{Z})} \sum_{m_i=1}^{\infty} \hat{\phi}_{(0, \dots, 0, m_i, 0, \dots, 0)} \left(\begin{pmatrix} \gamma_i & \\ & I_{n-i} \end{pmatrix} z \right) \right) \prod_{h=1}^{n-1} dx_{h,n} \end{aligned}$$

By the previous lemma, we see that only the term

$$\sum_{\gamma_i \in P_{(i-1,1)}(\mathbb{Z}) \setminus SL_i(\mathbb{Z})} \sum_{m_i=1}^{\infty} \hat{\phi}_{(0, \dots, 0, m_i, 0, \dots, 0)} \left(\begin{pmatrix} \gamma_i & \\ & I_{n-i} \end{pmatrix} z \right)$$

has the variables $x_{h,n}$ for $1 \leq h \leq n-1$ and these variables only appears in the exponential factor as $e \left(\sum_{j=1}^{n-1} \gamma_{n-1,j} x_{j,n} \right)$, where $\gamma_{a,b}$ are entries of the matrix γ_{n-1} . Thus

$$\int_{(\mathbb{Z} \setminus \mathbb{R})^{n-1}} \sum_{\gamma_i \in P_{(i-1,1)}(\mathbb{Z}) \setminus SL_i(\mathbb{Z})} \sum_{m_i=1}^{\infty} \hat{\phi}_{(0, \dots, 0, m_i, 0, \dots, 0)} \left(\begin{pmatrix} \gamma_i & \\ & I_{n-i} \end{pmatrix} z \right) \prod_{h=1}^{n-1} dx_{h,n} = 0.$$

So we have

$$\begin{aligned}
& \int_{(\mathbb{Z} \setminus \mathbb{R})^{n-1}} E_{(n-1,1)}(z, s, 1) \prod_{h=1}^{n-1} dx_{h,n} \\
&= \sum_{m_1 \in \mathbb{Z}} \hat{\phi}_{(m_1, 0, \dots, 0)}(z) + \sum_{i=2}^{n-2} \sum_{\gamma_i \in P_{(i-1,1)}(\mathbb{Z}) \setminus SL_i(\mathbb{Z})} \sum_{m_i=1}^{\infty} \hat{\phi}_{(0, \dots, 0, m_i, 0, \dots, 0)} \left(\begin{pmatrix} \gamma_i & \\ & I_{n-i} \end{pmatrix} z \right) \\
&= 2(y_1^{n-1} \cdots y_{n-1})^s + \frac{\xi(ns-1)}{\xi(ns)} (y_1^{n-1} \cdots y_{n-1})^{\frac{1-s}{n-1}} E_{(n-2,1)} \left(\mathbf{m}_{n-1}(z), \frac{ns-1}{n-1}, 1 \right)
\end{aligned}$$

by comparing Fourier expansions. □

Proposition 3.2.

$$\begin{aligned}
& \int_{(\mathbb{Z} \setminus \mathbb{R})^{n-1}} |E_{(n-1,1)}(z, s, 1)|^2 \prod_{h=1}^{n-1} dx_{h,n} \\
&= 4\ell^{-2na} + \frac{|\xi(ns-1)|^2}{|\xi(ns)|^2} (y_1^{n-1} \cdots y_{n-1})^{\frac{2-a}{n-1}} \left| E_{(n-2,1)} \left(\mathbf{m}_{n-1}(z), \frac{ns-1}{n-1}, 1 \right) \right|^2 \\
&+ 2\ell^{-\frac{n^2a-2na+n-in^2b}{n-1}} \frac{\xi(n\bar{s}-1)}{\xi(n\bar{s})} E_{(n-2,1)} \left(\mathbf{m}_{n-1}(z), \frac{n\bar{s}-1}{n-1}, 1 \right) \\
&+ 2\ell^{-\frac{n^2a-2na+n-in^2b}{n-1}} \frac{\xi(ns-1)}{\xi(ns)} E_{(n-2,1)} \left(\mathbf{m}_{n-1}(z), \frac{ns-1}{n-1}, 1 \right) \\
&+ \frac{8\ell^{-2na}}{|\xi(s)|^2} \sum_{m_{n-1}=1}^{\infty} \sum_{\gamma \in P_{(n-2,1)}(\mathbb{Z}) \setminus SL_{n-1}(\mathbb{Z})} m_{n-1}^{na-1} \sigma_{-na+1-inb}(m_{n-1}) \sigma_{-na+1+inb}(m_{n-1}) \left(\frac{\ell^{-\frac{n}{n-1}}}{\det(\mathbf{m}_{n-1}(z))^{\frac{1}{n-1}}} \right)^{-na+1} \\
&\times K_{\frac{na-1}{2} + \frac{inb}{2}} \left(2\pi m_{n-1} \frac{\ell^{-\frac{n}{n-1}}}{\det(\mathbf{m}_{n-1}(z))^{\frac{1}{n-1}}} \right) K_{\frac{na-1}{2} - \frac{inb}{2}} \left(2\pi m_{n-1} \frac{\ell^{-\frac{n}{n-1}}}{\det(\mathbf{m}_{n-1}(z))^{\frac{1}{n-1}}} \right) \Big|_{\gamma}
\end{aligned}$$

where $s = a + bi$, $\ell^{-n} = y_1^{n-1} \cdots y_{n-1}$ and $\det(\mathbf{m}_{n-1}(z)) = y_2^{n-2} \cdots y_{n-1}$.

Proof. By the previous lemma we can open the square and find

$$\begin{aligned}
& \int_{(\mathbb{Z} \setminus \mathbb{R})^{n-1}} |E_{(n-1,1)}(z, s, 1)|^2 \prod_{h=1}^{n-1} dx_{h,n} \\
&= \left| 2 (y_1^{n-1} \cdots y_{n-1})^s + \frac{\xi(ns-1)}{\xi(ns)} (y_1^{n-1} \cdots y_{n-1})^{\frac{1-s}{n-1}} E_{(n-2,1)} \left(\mathbf{m}_{n-1}(z), \frac{ns-1}{n-1}, 1 \right) \right|^2 \\
&+ \frac{8 (y_1^{n-1} \cdots y_{n-1})^{2a}}{|\xi(s)|^2} \sum_{m_{n-1}=1}^{\infty} \sum_{\gamma \in P_{(n-2,1)}(\mathbb{Z}) \setminus SL_{n-1}(\mathbb{Z})} m_{n-1}^{na-1} \sigma_{-na+1-inb}(m_{n-1}) \sigma_{-na+1+inb}(m_{n-1}) \\
&\times K_{\frac{na-1}{2} + \frac{inb}{2}}(2\pi m_{n-1} y_1) K_{\frac{na-1}{2} - \frac{inb}{2}}(2\pi m_{n-1} y_1) y_1^{-na+1} \Big|_{\gamma} \\
&= 4 (y_1^{n-1} \cdots y_{n-1})^{2a} + \frac{|\xi(ns-1)|^2}{|\xi(ns)|^2} (y_1^{n-1} \cdots y_{n-1})^{\frac{2-a}{n-1}} \left| E_{(n-2,1)} \left(\mathbf{m}_{n-1}(z), \frac{ns-1}{n-1}, 1 \right) \right|^2 \\
&+ 2 (y_1^{n-1} \cdots y_{n-1})^{\frac{na-2a+1+inb}{n-1}} \frac{\xi(n\bar{s}-1)}{\xi(n\bar{s})} E_{(n-2,1)} \left(\mathbf{m}_{n-1}(z), \frac{n\bar{s}-1}{n-1}, 1 \right) \\
&+ 2 (y_1^{n-1} \cdots y_{n-1})^{\frac{na-2a+1-inb}{n-1}} \frac{\xi(ns-1)}{\xi(ns)} E_{(n-2,1)} \left(\mathbf{m}_{n-1}(z), \frac{ns-1}{n-1}, 1 \right) \\
&+ \frac{8 (y_1^{n-1} \cdots y_{n-1})^{2a}}{|\xi(s)|^2} \sum_{m_{n-1}=1}^{\infty} \sum_{\gamma \in P_{(n-2,1)}(\mathbb{Z}) \setminus SL_{n-1}(\mathbb{Z})} m_{n-1}^{na-1} \sigma_{-na+1-inb}(m_{n-1}) \sigma_{-na+1+inb}(m_{n-1}) \\
&\times K_{\frac{na-1}{2} + \frac{inb}{2}}(2\pi m_{n-1} y_1) K_{\frac{na-1}{2} - \frac{inb}{2}}(2\pi m_{n-1} y_1) y_1^{-na+1} \Big|_{\gamma}
\end{aligned}$$

We get our desired result after the change of variable $\ell^{-n} = y_1^{n-1} \cdots y_{n-1}$ and $\det(\mathbf{m}_{n-1}(z)) = y_2^{n-2} \cdots y_{n-1}$. \square

4. CUSPIDAL AND NON-MINIMAL EISENSTEIN CONTRIBUTION

In this section we show that

(1)

$$\int_{SL_n(\mathbb{Z}) \backslash X_n} \phi(z) \left| E_{n-1,1} \left(z, \frac{1}{2} + it, 1 \right) \right|^2 d^*z = 0,$$

for $\phi(z)$ a cusp form on $GL(n)$ with $n \geq 3$.

(2)

$$\int_{SL_n(\mathbb{Z}) \backslash X_n} E_{(n_1, \dots, n_r)}(z, \psi, u_1, \dots, u_r) \left| E_{n-1,1} \left(z, \frac{1}{2} + it, 1 \right) \right|^2 d^*z = 0$$

if $n \geq 3$ and the partition (n_1, \dots, n_r) is not of type $(2, 1, \dots, 1)$.

(3)

$$\int_{SL_n(\mathbb{Z}) \backslash X_n} E_{(2,1,\dots,1)}(z, \psi, \phi) \left| E_{n-1,1} \left(z, \frac{1}{2} + it, 1 \right) \right|^2 d^*z = O_\epsilon(t^{-\frac{1}{2}+\epsilon}).$$

4.1. Some Basic Lemmata. Here we write down some basic lemmata that we will use repeatedly in the rest of the paper.

Lemma 4.1. (*Stade's formula for $GL(2)$*)

The Mellin transform of product of K -Bessel functions can be evaluated (6.576.4 of [GR])

$$(4.1) \quad \int_0^\infty K_\mu(y) K_\nu(y) y^s \frac{dy}{y} = 2^{s-3} \frac{\Gamma\left(\frac{s+\mu+\nu}{2}\right) \Gamma\left(\frac{s+\mu-\nu}{2}\right) \Gamma\left(\frac{s-\mu+\nu}{2}\right) \Gamma\left(\frac{s-\mu-\nu}{2}\right)}{\Gamma(s)}.$$

Lemma 4.2. We have the following beautiful formula given by Ramanujan [Ra].

$$(4.2) \quad \sum_{n=1}^\infty \frac{\sigma_a(n) \sigma_b(n)}{n^s} = \frac{\zeta(s) \zeta(s-a) \zeta(s-b) \zeta(s-a-b)}{\zeta(2s-a-b)}$$

for $\text{Re}(s)$ large enough.

Lemma 4.3. (*Mellin inversion*) (See 3.1.1 of [PK])

The Mellin transform of a absolutely integrable function $f(x)$ on $(0, \infty)$ is defined by

$$F(s) = \int_0^\infty x^{s-1} f(x) dx$$

when the integral converges. If we suppose for small $\epsilon > 0$

$$f(x) = \begin{cases} O(x^{-a-\epsilon}) & \text{as } x \rightarrow 0^+ \\ O(x^{-b+\epsilon}) & \text{as } x \rightarrow \infty, \end{cases}$$

then for $a < c < b$, $f(x)$ can be recovered by Mellin inversion:

$$f(x) = \frac{1}{2\pi i} \int_{(c)} x^{-s} F(s) ds.$$

4.2. Partition of the Form $n = n_1 + \cdots + n_{r-1} + 1$. We now fix a partition $n = n_1 + \cdots + n_{r-1} + 1 \neq 1 + \cdots + 1$ in non-increasing order. Our goal this section is to reduce the calculation involving Eisenstein series associated to the parabolic subgroup $P_{n_1, \dots, n_{r-1}, 1}$ to the calculation involving Eisenstein series associated to the parabolic subgroup $P_{n_1, \dots, n_{r-1}}$. This argument can be repeated so that we are left with partitions of the form $n = n_1 + \cdots + n_r$ with $n_r \geq 2$, which will be tackled in the next section.

First we have the following rearrangement of Eisenstein series

Lemma 4.4.

$$(4.3) \quad \begin{aligned} & E_{(n_1, \dots, n_{r-1}, 1)}(z, s, \phi_1, \dots, \phi_{r-1}, 1) \\ &= E_{(n-1, 1)}(z, s'', E_{(n_1, \dots, n_{r-1})}(\mathbf{m}_{n-1}(z), s', \phi_1, \dots, \phi_{r-1}), 1) \end{aligned}$$

where $s' = (s'_1, \dots, s'_{r-1}) \in \mathbb{C}^{r-1}$ and $s'' \in \mathbb{C}$ are given by

$$s'_i = s_i - \frac{n_1 s_1 + \cdots + n_{r-1} s_{r-1}}{n-1}, \quad s'' = \frac{n_1 s_1 + \cdots + n_{r-1} s_{r-1}}{n-1}.$$

Proof. By definition,

$$\begin{aligned} & E_{(n_1, \dots, n_{r-1}, 1)}(z, s, \phi_1, \dots, \phi_{r-1}, 1) \\ &= \sum_{\gamma \in P_{n_1, \dots, n_{r-1}, 1}(\mathbb{Z}) \backslash SL_n(\mathbb{Z})} \prod_{i=1}^{r-1} \phi_i(\mathbf{m}_i(\gamma z)) I_s(\gamma z, P_{n_1, \dots, n_{r-1}, 1}) \\ &= \sum_{\beta \in P_{n_1, \dots, n_{r-1}, 1}(\mathbb{Z}) \backslash P_{n-1, 1}(\mathbb{Z})} \sum_{\alpha \in P_{n-1, 1}(\mathbb{Z}) \backslash SL_n(\mathbb{Z})} \prod_{i=1}^{r-1} \phi_i(\mathbf{m}_i(\beta \alpha z)) I_s(\beta \alpha z, P_{n_1, \dots, n_{r-1}, 1}) \end{aligned}$$

Here we can write

$$I_s(z, P_{n_1, \dots, n_{r-1}, 1}) = I_{s'}(\mathbf{m}_{n-1}(z), P_{n_1, \dots, n_{r-1}}) (y_1^{n-1} y_2^{n-2} \cdots y_{n-1})^{s''}.$$

where $s' = (s'_1, \dots, s'_{r-1}) \in \mathbb{C}^{r-1}$ and $s'' \in \mathbb{C}$ are given by

$$s'_i = s_i - \frac{n_1 s_1 + \cdots + n_{r-1} s_{r-1}}{n-1}, \quad s'' = \frac{n_1 s_1 + \cdots + n_{r-1} s_{r-1}}{n-1}.$$

One easily can check that $n_1 s'_1 + \cdots + n_{r-1} s'_{r-1} = 0$ so that $I_{s'}(\mathbf{m}_{n-1}(z), P_{n_1, \dots, n_{r-1}})$ is a well defined I -function with spectral parameter s' . Thus we have

$$\begin{aligned} & E_{(n_1, \dots, n_{r-1}, 1)}(z, s, \phi_1, \dots, \phi_{r-1}, 1) \\ &= \sum_{\beta \in P_{n_1, \dots, n_{r-1}, 1}(\mathbb{Z}) \backslash P_{n-1, 1}(\mathbb{Z})} \sum_{\alpha \in P_{n-1, 1}(\mathbb{Z}) \backslash SL_n(\mathbb{Z})} \prod_{i=1}^{r-1} \phi_i(\mathbf{m}_i(z)) I_{s'}(\mathbf{m}_{n-1}(z), P_{n_1, \dots, n_{r-1}}) (y_1^{n-1} y_2^{n-2} \cdots y_{n-1})^{s''} \Big|_{\beta} \Big|_{\alpha} \\ &= \sum_{\alpha \in P_{n-1, 1}(\mathbb{Z}) \backslash SL_n(\mathbb{Z})} \left(\sum_{\beta \in P_{n_1, \dots, n_{r-1}, 1}(\mathbb{Z}) \backslash P_{n-1, 1}(\mathbb{Z})} \prod_{i=1}^{r-1} \phi_i(\mathbf{m}_i(\beta z)) I_{s'}(\mathbf{m}_{n-1}(\beta z), P_{n_1, \dots, n_{r-1}}) \right) (y_1^{n-1} y_2^{n-2} \cdots y_{n-1})^{s''} \Big|_{\alpha} \\ &= \sum_{\alpha \in P_{n-1, 1}(\mathbb{Z}) \backslash SL_n(\mathbb{Z})} \left(\sum_{\beta \in P_{n_1, \dots, n_{r-1}}(\mathbb{Z}) \backslash SL_{n-1}(\mathbb{Z})} \prod_{i=1}^{r-1} \phi_i(\mathbf{m}_i(\beta z)) I_{s'}(\mathbf{m}_{n-1}(\beta z), P_{n_1, \dots, n_{r-1}}) \right) (y_1^{n-1} y_2^{n-2} \cdots y_{n-1})^{s''} \Big|_{\alpha} \\ &= \sum_{\alpha \in P_{n-1, 1}(\mathbb{Z}) \backslash SL_n(\mathbb{Z})} E_{(n_1, \dots, n_{r-1})}(\mathbf{m}_{n-1}(z), s', \phi_1, \dots, \phi_{r-1}) (y_1^{n-1} y_2^{n-2} \cdots y_{n-1})^{s''} \Big|_{\alpha} \\ &= E_{(n-1, 1)}(z, s'', E_{(n_1, \dots, n_{r-1})}(\mathbf{m}_{n-1}(z), s', \phi_1, \dots, \phi_{r-1}), 1) \end{aligned}$$

as $(y_1^{n-1} y_2^{n-2} \cdots y_{n-1})$ is invariant under the action of β by Lemma 3.3. \square

Lemma 4.5. *We have the decomposition of measurable space*

$$(4.4) \quad P_{n-1,1}(\mathbb{Z}) \backslash X_n = SL_{n-1}(\mathbb{Z}) \backslash X_{n-1} \times (\mathbb{R}/\mathbb{Z})^{n-1} \times \mathbb{R}^+.$$

with

$$d^* z = -\frac{1}{\xi(n)} d^* z' \prod_{h=1}^{n-1} dx_{h,n} \ell^n \frac{d\ell}{\ell}.$$

where

$$\ell = \left(\prod_{i=1}^{n-1} y_i^{n-i} \right)^{-\frac{1}{n}} \quad z' = \mathfrak{m}_{n-1}(z).$$

Proof. By matrix multiplication, we have

$$\begin{aligned} z = X \cdot Y &= \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \cdots & x_{1,n} \\ & 1 & x_{2,3} & \cdots & x_{2,n} \\ & & \ddots & & \vdots \\ & & & 1 & x_{n-1,n} \\ & & & & 1 \end{pmatrix} \begin{pmatrix} \ell & y_1 y_2 \cdots y_{n-1} & & & \\ & \ell & y_1 y_2 \cdots y_{n-2} & & \\ & & \ddots & & \\ & & & \ell & y_1 \\ & & & & \ell \end{pmatrix} \\ &= \begin{pmatrix} 1 & & & x_{1,n} \\ & 1 & & x_{2,n} \\ & & \ddots & \vdots \\ & & & 1 & x_{n-1,n} \\ & & & & 1 \end{pmatrix} \cdot \begin{pmatrix} z' & \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} \ell^{-\frac{1}{n-1}} \cdot I_{n-1} & \\ & \ell \end{pmatrix} \\ &= \begin{pmatrix} I_{n-1} & \bar{x}_n \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} z' & \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} \ell^{-\frac{1}{n-1}} \cdot I_{n-1} & \\ & \ell \end{pmatrix}, \end{aligned}$$

where

$$z' = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \cdots & x_{1,n-1} \\ & 1 & x_{2,3} & \cdots & x_{2,n-1} \\ & & \ddots & & \vdots \\ & & & 1 & x_{n-2,n-1} \\ & & & & 1 \end{pmatrix} \begin{pmatrix} \ell^{\frac{n}{n-1}} & y_1 y_2 \cdots y_{n-1} & & & \\ & \ell^{\frac{n}{n-1}} & y_1 y_2 \cdots y_{n-2} & & \\ & & \ddots & & \\ & & & \ell^{\frac{n}{n-1}} & y_1 \end{pmatrix} = X' \cdot Y'.$$

By direct computation, we have for $\gamma \in SL_{n-1}(\mathbb{Z})$

$$\begin{aligned} &\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} \cdot z \\ &= \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} I_{n-1} & \bar{x}_n \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} z' & \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} \ell^{-\frac{1}{n-1}} \cdot I_{n-1} & \\ & \ell \end{pmatrix} \\ &= \begin{pmatrix} \gamma & \gamma \bar{x}_n \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} z' & \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} \ell^{-\frac{1}{n-1}} \cdot I_{n-1} & \\ & \ell \end{pmatrix} \\ &= \begin{pmatrix} \gamma & \gamma \bar{x}_n \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} \gamma^{-1} & \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} z' & \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} \ell^{-\frac{1}{n-1}} \cdot I_{n-1} & \\ & \ell \end{pmatrix} \\ &= \begin{pmatrix} I_{n-1} & \gamma \bar{x}_n \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} \gamma z' & \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} \ell^{-\frac{1}{n-1}} \cdot I_{n-1} & \\ & \ell \end{pmatrix}. \end{aligned}$$

This gives the decomposition

$$P_{n-1,1}(\mathbb{Z}) \backslash X_n = SL_{n-1}(\mathbb{Z}) \backslash X_{n-1} \times (\mathbb{R}/\mathbb{Z})^{n-1} \times \mathbb{R}^+.$$

Let $F(z)$ be an automorphic function. We can unfold the following product:

$$\begin{aligned} & \int_{SL_n \backslash X_n} F(z) E_{n-1,1}(z, 1, s) d^* z \\ &= \int_{P_{n-1,1} \backslash X_n} F(z) \det(Y)^s d^* z \\ &= \int_{SL_{n-1}(\mathbb{Z}) \backslash X_{n-1}} \int_0^\infty \int_{(\mathbb{Z}/\mathbb{R})^{n-1}} F(\bar{x}_n, z', \ell) \ell^{-ns} \left(-\frac{n}{n-1} \right) d\bar{x}_n \ell^n \frac{d\ell}{\ell} d^* z' \end{aligned}$$

Now define

$$G(z') := \int_0^\infty \int_{(\mathbb{Z}/\mathbb{R})^{n-1}} F(\bar{x}_n, z', \ell) \ell^{-ns} \left(-\frac{n}{n-1} \right) d\bar{x}_n \ell^n \frac{d\ell}{\ell}$$

We claim that $G(z')$ is automorphic.

$$G(\gamma z') = \int_0^\infty \int_{(\mathbb{Z}/\mathbb{R})^{n-1}} F(\bar{x}_n, \gamma z', \ell) \ell^{-ns} \left(-\frac{n}{n-1} \right) d\bar{x}_n \ell^n \frac{d\ell}{\ell}$$

With the fact that F is automorphic we have

$$\begin{aligned} & F(\bar{x}_n, \gamma z', \ell) \\ &= F \left(\begin{pmatrix} I_{n-1} & \bar{x}_n \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} \gamma z' & \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} \ell^{-\frac{1}{n-1}} \cdot I_{n-1} & \\ & \ell \end{pmatrix} \right) \\ &= F \left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} \begin{pmatrix} I_{n-1} & \gamma^{-1} \bar{x}_n \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} z' & \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} \ell^{-\frac{1}{n-1}} \cdot I_{n-1} & \\ & \ell \end{pmatrix} \right) \\ &= F \left(\begin{pmatrix} I_{n-1} & \gamma^{-1} \bar{x}_n \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} z' & \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} \ell^{-\frac{1}{n-1}} \cdot I_{n-1} & \\ & \ell \end{pmatrix} \right). \end{aligned}$$

Now since $\gamma \in SL_{n-1}(\mathbb{Z})$ and $F(z)$ is automorphic, the integral over $(\mathbb{Z}/\mathbb{R})^{n-1}$ is invariant under the action of γ on \bar{x}_n . So

$$\begin{aligned} G(\gamma z') &= \int_0^\infty \int_{(\mathbb{Z}/\mathbb{R})^{n-1}} F(\bar{x}_n, \gamma z', \ell) \ell^{-ns} \left(-\frac{n}{n-1} \right) d\bar{x}_n \ell^n \frac{d\ell}{\ell} \\ &= \int_0^\infty \int_{(\mathbb{Z}/\mathbb{R})^{n-1}} F(\gamma^{-1} \bar{x}_n, z', \ell) \ell^{-ns} \left(-\frac{n}{n-1} \right) d\bar{x}_n \ell^n \frac{d\ell}{\ell} \\ &= \int_0^\infty \int_{(\mathbb{Z}/\mathbb{R})^{n-1}} F(\bar{x}_n, z', \ell) \ell^{-ns} \left(-\frac{n}{n-1} \right) d\bar{x}_n \ell^n \frac{d\ell}{\ell} \\ &= G(z'). \end{aligned}$$

□

Now let $\nu = a + bi$. By the definition of incomplete Eisenstein series and Lemma 4.5 we have

$$\begin{aligned}
& \int_{SL_n(\mathbb{Z}) \backslash X_n} E_{(n_1, \dots, n_{r-1}, 1)}(z, \eta, \phi_1, \dots, \phi_{r-1}, 1) \left| E_{(n-1, 1)}(z, \nu, 1) \right|^2 d^* z \\
&= \int_{SL_n(\mathbb{Z}) \backslash X_n} \frac{1}{(2\pi i)^{r-1}} \int_{(2)} \cdots \int_{(2)} \tilde{\eta}(s_1, \dots, s_{r-1}) E_{(n_1, \dots, n_{r-1}, 1)}(z, s, \phi_1, \dots, \phi_{r-1}, 1) ds_1 \cdots ds_{r-1} \\
&\quad \times \left| E_{(n-1, 1)}(z, \nu, 1) \right|^2 d^* z \\
&= -\frac{1}{\xi(n)} \int_{SL_{n-1}(\mathbb{Z}) \backslash X_{n-1}} \int_0^\infty \int_{(\mathbb{Z}/\mathbb{R})^{n-1}} \frac{1}{(2\pi i)^{r-1}} \int_{(2)} \cdots \int_{(2)} \tilde{\eta}(s_1, \dots, s_{r-1}) \\
&\quad E_{(n_1, \dots, n_{r-1})}(\mathbf{m}_{n-1}(z), s', \phi_1, \dots, \phi_{r-1}) \ell^{-ns''} \left| E_{(n-1, 1)}(z, \nu, 1) \right|^2 ds_1 \cdots ds_{r-1} d^* z' \prod_{h=1}^{n-1} dx_{h,n} \ell^n \frac{d\ell}{\ell}
\end{aligned}$$

where $s' = (s'_1, \dots, s'_{r-1}) \in \mathbb{C}^{r-1}$ and $s'' \in \mathbb{C}$ are given by

$$s'_i = s_i - \frac{n_1 s_1 + \cdots + n_{r-1} s_{r-1}}{n-1}, \quad s'' = \frac{n_1 s_1 + \cdots + n_{r-1} s_{r-1}}{n-1}.$$

as before.

Now by Proposition 3.2 we need to understand the following five integrals:

$$\begin{aligned}
\Xi_1 &= \frac{-1}{\xi(n)} \int_{SL_{n-1}(\mathbb{Z}) \backslash X_{n-1}} \int_0^\infty \frac{1}{(2\pi i)^{r-1}} \int_{(2)} \cdots \int_{(2)} \tilde{\eta}(s) E_{(n_1, \dots, n_{r-1})}(z', s', \phi_1, \dots, \phi_{r-1}) \ell^{-ns''} \\
&\quad \times 4\ell^{-2na} ds_1 \cdots ds_{r-1} d^* z' \ell^n \frac{d\ell}{\ell},
\end{aligned}$$

$$\begin{aligned}
\Xi_2 &= \frac{-1}{\xi(n)} \int_{SL_{n-1}(\mathbb{Z}) \backslash X_{n-1}} \int_0^\infty \frac{1}{(2\pi i)^{r-1}} \int_{(2)} \cdots \int_{(2)} \tilde{\eta}(s) E_{(n_1, \dots, n_{r-1})}(z', s', \phi_1, \dots, \phi_{r-1}) \ell^{-ns''} \\
&\quad \times \frac{|\xi(n\nu - 1)|^2}{|\xi(n\nu)|^2} \ell^{-n \frac{2-a}{n-1}} \left| E_{(n-2, 1)}\left(z', \frac{n\nu - 1}{n-1}, 1\right) \right|^2 ds_1 \cdots ds_{r-1} d^* z' \ell^n \frac{d\ell}{\ell},
\end{aligned}$$

$$\begin{aligned}
\Xi_3 &= \frac{-1}{\xi(n)} \int_{SL_{n-1}(\mathbb{Z}) \backslash X_{n-1}} \int_0^\infty \frac{1}{(2\pi i)^{r-1}} \int_{(2)} \cdots \int_{(2)} \tilde{\eta}(s) E_{(n_1, \dots, n_{r-1})}(z', s', \phi_1, \dots, \phi_{r-1}) \ell^{-ns''} \\
&\quad \times \ell^{-\frac{n^2 a - 2na + n + in^2 b}{n-1}} \frac{\xi(n\bar{\nu} - 1)}{\xi(n\bar{\nu})} E_{(n-2, 1)}\left(z', \frac{n\bar{\nu} - 1}{n-1}, 1\right) ds_1 \cdots ds_{r-1} d^* z' \ell^n \frac{d\ell}{\ell},
\end{aligned}$$

$$\begin{aligned}
\Xi'_3 &= \frac{-1}{\xi(n)} \int_{SL_{n-1}(\mathbb{Z}) \backslash X_{n-1}} \int_0^\infty \frac{1}{(2\pi i)^{r-1}} \int_{(2)} \cdots \int_{(2)} \tilde{\eta}(s) E_{(n_1, \dots, n_{r-1})}(z', s', \phi_1, \dots, \phi_{r-1}) \ell^{-ns''} \\
&\quad \times \ell^{-\frac{n^2 a - 2na + n - in^2 b}{n-1}} \frac{\xi(n\nu - 1)}{\xi(n\nu)} E_{(n-2, 1)}\left(z', \frac{n\nu - 1}{n-1}, 1\right) ds_1 \cdots ds_{r-1} d^* z' \ell^n \frac{d\ell}{\ell},
\end{aligned}$$

$$\begin{aligned}
\Xi_4 &= \frac{-1}{\xi(n)} \int_{SL_{n-1}(\mathbb{Z}) \backslash X_{n-1}} \int_0^\infty \frac{1}{(2\pi i)^{r-1}} \int_{(2)} \cdots \int_{(2)} \tilde{\eta}(s) E_{(n_1, \dots, n_{r-1})}(z', s', \phi_1, \dots, \phi_{r-1}) \ell^{-ns''} \\
&\quad \times \frac{8\ell^{-2na}}{|\xi(\nu)|^2} \sum_{m_{n-1}=1}^\infty \sum_{\gamma \in P_{(n-2,1)}(\mathbb{Z}) \backslash SL_{n-1}(\mathbb{Z})} m_{n-1}^{na-1} \sigma_{-na+1-inb}(m_{n-1}) \sigma_{-na+1+inb}(m_{n-1}) \left(\frac{\ell^{-\frac{n}{n-1}}}{\det(z')^{\frac{1}{n-1}}} \right)^{-na+1} \\
&\quad \times K_{\frac{na-1}{2} + \frac{inb}{2}} \left(2\pi m_{n-1} \frac{\ell^{-\frac{n}{n-1}}}{\det(z')^{\frac{1}{n-1}}} \right) K_{\frac{na-1}{2} - \frac{inb}{2}} \left(2\pi m_{n-1} \frac{\ell^{-\frac{n}{n-1}}}{\det(z')^{\frac{1}{n-1}}} \right) \Big|_\gamma ds_1 \cdots ds_{r-1} d^* z' \ell^n \frac{d\ell}{\ell}.
\end{aligned}$$

Lemma 4.6. For $(n_1, \dots, n_{r-1}, 1) \neq (1, \dots, 1)$, we have that $\Xi_1 = 0$.

Proof. Since $(n_1, \dots, n_{r-1}, 1) \neq (1, \dots, 1)$, the constant term of $E_{(n_1, \dots, n_{r-1})}(z', s', \phi_1, \dots, \phi_{r-1})$ along the minimal parabolic is 0. So we are integrating an automorphic function without constant term over the entire quotient $SL_{n-1}(\mathbb{Z}) \backslash X_{n-1}$. We must have $\Xi_1 = 0$. \square

Lemma 4.7. For $(n_1, \dots, n_{r-1}, 1) \neq (1, \dots, 1)$, we have that $\Xi_3 = \Xi'_3 = 0$.

Proof. We can rearrange Ξ_3 and get

$$\begin{aligned}
\Xi_3 &= \frac{-1}{\xi(n)} \frac{\xi(n\bar{\nu} - 1)}{\xi(n\bar{\nu})} \int_{SL_{n-1}(\mathbb{Z}) \backslash X_{n-1}} \int_0^\infty \frac{1}{(2\pi i)^{r-1}} \int_{(2)} \cdots \int_{(2)} \tilde{\eta}(s) E_{(n_1, \dots, n_{r-1})}(z', s', \phi_1, \dots, \phi_{r-1}) \ell^{-ns''} \\
&\quad \times \ell^{-\frac{n^2 a - 2na + n + in^2 b}{n-1}} E_{(n-2,1)} \left(z', \frac{n\bar{\nu} - 1}{n-1}, 1 \right) ds_1 \cdots ds_{r-1} d^* z' \ell^n \frac{d\ell}{\ell}
\end{aligned}$$

We can now apply the Mellin inversion theorem to eliminate the integral over ℓ . Thus Ξ_3 has the shape of

$$\begin{aligned}
\Xi_3 &= c \int_{SL_{n-1}(\mathbb{Z}) \backslash X_{n-1}} \frac{1}{(2\pi i)^{r-2}} \int_{(2)} \cdots \int_{(2)} f(s') E_{(n_1, \dots, n_{r-1})}(z', s', \phi_1, \dots, \phi_{r-1}) \\
&\quad \times E_{(n-2,1)}(z', \nu', 1) ds'_1 \cdots ds'_{r-2} d^* z'
\end{aligned}$$

for some f with rapid decay in the imaginary parts of the arguments. At this point, if $n_r \geq 2$, then unfolding with respect to $E_{(n-2,1)}(z', \nu', 1)$ and applying Proposition 2.2 reduce Ξ_3 to 0. If $n_r = 1$, we unfold with respect to $E_{(n_1, \dots, n_{r-1})}(z', s', \phi_1, \dots, \phi_{r-1})$ and apply 3.1 repeatedly until the last partition number is no longer 1 and apply Proposition 2.2. \square

Lemma 4.8. For $(n_1, \dots, n_{r-1}, 1) \neq (1, \dots, 1)$, we have that $\Xi_4 = 0$.

Proof.

$$\begin{aligned}
\Xi_4 &= \frac{-1}{\xi(n)} \int_{SL_{n-1}(\mathbb{Z}) \backslash X_{n-1}} \int_0^\infty \frac{1}{(2\pi i)^{r-1}} \int_{(2)} \cdots \int_{(2)} \tilde{\eta}(s) E_{(n_1, \dots, n_{r-1})}(z', s', \phi_1, \dots, \phi_{r-1}) \ell^{-ns''} \\
&\quad \times \frac{8\ell^{-2na}}{|\xi(\nu)|^2} \sum_{m_{n-1}=1}^\infty \sum_{\gamma \in P_{(n-2,1)}(\mathbb{Z}) \backslash SL_{n-1}(\mathbb{Z})} m_{n-1}^{na-1} \sigma_{-na+1-inb}(m_{n-1}) \sigma_{-na+1+inb}(m_{n-1}) \left(\frac{\ell^{-\frac{n}{n-1}}}{\det(z')^{\frac{1}{n-1}}} \right)^{-na+1} \\
&\quad \times K_{\frac{na-1}{2} + \frac{inb}{2}} \left(2\pi m_{n-1} \frac{\ell^{-\frac{n}{n-1}}}{\det(z')^{\frac{1}{n-1}}} \right) K_{\frac{na-1}{2} - \frac{inb}{2}} \left(2\pi m_{n-1} \frac{\ell^{-\frac{n}{n-1}}}{\det(z')^{\frac{1}{n-1}}} \right) \Big|_\gamma ds_1 \cdots ds_{r-1} d^* z' \ell^n \frac{d\ell}{\ell}.
\end{aligned}$$

Recall that $\frac{\ell^{-\frac{n}{n-1}}}{\det(z')^{\frac{1}{n-1}}} = y_1$ and the effect of γ on y_1 is given by part (3) of Lemma 3.3. We have $y'_1 = y_1 (b_1^2 + \cdots + b_{n-1}^2)^{\frac{1}{2}}$ where b'_i s are defined in Lemma 3.3. We can then make the change of variable

$m_{n-1} \frac{\ell^{-\frac{n}{n-1}}}{\det(z')^{\frac{n}{n-1}}} (b_1^2 + \dots + b_{n-1}^2)^{\frac{1}{2}} \mapsto u$. The sum of $(b_1^2 + \dots + b_{n-1}^2)^{\frac{1}{2}}$ over $P_{(n-2,1)}(\mathbb{Z}) \backslash SL_{n-1}(\mathbb{Z})$ is simply a degenerate Eisenstein series. The sum over m_{n-1} of divisor functions is a ratio of Zeta functions by Lemma 4.2. This together with ratio of Gamma functions (formed after integrating over u of the K -Bessel functions) will give completed Zeta functions. We can now apply the previous Lemma which shows that $\Xi_4 = 0$. \square

Proposition 4.1. *For $(n_1, \dots, n_{r-1}, 1) \neq (1, \dots, 1)$, we have that*

$$\begin{aligned} & \int_{SL_n(\mathbb{Z}) \backslash X_n} E_{(n_1, \dots, n_{r-1}, 1)}(z, \eta, \phi_1, \dots, \phi_{r-1}, 1) |E_{(n-1, 1)}(z, \nu, 1)|^2 d^* z \\ &= C \frac{|\xi(n\nu - 1)|^2}{|\xi(n\nu)|^2} \int_{SL_{n-1}(\mathbb{Z}) \backslash X_{n-1}} \frac{1}{(2\pi i)^{r-2}} \int_{(2)} \dots \int_{(2)} \psi(s') E_{(n_1, \dots, n_{r-1})}(z', s', \phi_1, \dots, \phi_{r-1}) \\ & \quad \times \left| E_{(n-2, 1)}\left(z', \frac{n\nu - 1}{n - 1}, 1\right) \right|^2 ds'_1 \dots ds'_{r-2} d^* z' \end{aligned}$$

for some absolute constant C .

Proof. From the previous three lemmas, we see that the only term left is Ξ_2 . We have the following simplification for Ξ_2 :

$$\begin{aligned} \Xi_2 &= \frac{-1}{\xi(n)} \int_{SL_{n-1}(\mathbb{Z}) \backslash X_{n-1}} \int_0^\infty \frac{1}{(2\pi i)^{r-1}} \int_{(2)} \dots \int_{(2)} \tilde{\eta}(s) E_{(n_1, \dots, n_{r-1})}(z', s', \phi_1, \dots, \phi_{r-1}) \ell^{-ns''} \\ & \quad \times \frac{|\xi(n\nu - 1)|^2}{|\xi(n\nu)|^2} \ell^{-n \frac{2-a}{n-1}} \left| E_{(n-2, 1)}\left(z', \frac{n\nu - 1}{n - 1}, 1\right) \right|^2 ds_1 \dots ds_{r-1} d^* z' \ell^n \frac{d\ell}{\ell} \\ &= -\frac{n^2 - n}{n_{r-1}} \frac{|\xi(n\nu - 1)|^2}{\xi(n) |\xi(n\nu)|^2} \int_{SL_{n-1}(\mathbb{Z}) \backslash X_{n-1}} \frac{1}{(2\pi i)^{r-2}} \int_{(2)} \dots \int_{(2)} \psi(s') E_{(n_1, \dots, n_{r-1})}(z', s', \phi_1, \dots, \phi_{r-1}) \\ & \quad \times \left| E_{(n-2, 1)}\left(z', \frac{n\nu - 1}{n - 1}, 1\right) \right|^2 ds'_1 \dots ds'_{r-2} d^* z', \end{aligned}$$

where the factor $\frac{n^2 - n}{n_{r-1}}$ comes from the Jacobian of the change of variables and

$$\psi(s') := \tilde{\eta}(s'_1 + c, \dots, s'_{r-2} + c, -(s'_1 + \dots + s'_{r-2} + c))$$

where $c = \frac{2-a}{n-1} - 1$. \square

This proposition can be used until we reach the following:

(1)

$$\int_{SL_2(\mathbb{Z}) \backslash \mathbb{H}} \phi(z) |E(z, s)|^2 d^* z$$

where ϕ is a cusp form on $GL(2)$ and $E(z, s)$ is the Eisenstein series on $GL(2)$.

(2)

$$\int_{SL_n(\mathbb{Z}) \backslash X_n} \phi(z) |E_{n-1, 1}(z, s, 1)|^2 d^* z$$

where ϕ is a cusp form on $GL(n)$ for $n \geq 3$.

(3)

$$\int_{SL_n(\mathbb{Z}) \backslash X_n} E_{(n_1, \dots, n_r)}(z, \eta, \phi_1, \dots, \phi_r) |E_{(n-1, 1)}(z, \nu, 1)|^2 d^* z$$

for $n = n_1 + \dots + n_r$ in non-increasing order with $n_r \geq 2$.

We will consider these three separately in the next three sections.

4.3. A $GL(2)$ Calculation. In this section, our goal is to show the following estimate.

Theorem 4.1.

$$\int_{SL_n(\mathbb{Z}) \backslash X_n} E_{(2,1,\dots,1)}(z, \eta, \phi) \left| E_{n-1,1} \left(z, \frac{1}{2} + it, 1 \right) \right|^2 d^*z \ll t^{-\frac{1}{2}+\epsilon}.$$

Remark 4.1. We get this rate by only utilizing the convexity bounds for the Zeta function and L-functions associated to Maass forms.

Proof. First we apply Proposition 4.1 ($n-2$) times and get

$$\begin{aligned} & \int_{SL_n(\mathbb{Z}) \backslash X_n} E_{(2,1,\dots,1)}(z, \eta, \phi) \left| E_{n-1,1} \left(z, \frac{1}{2} + it, 1 \right) \right|^2 d^*z \\ & \ll \frac{|\xi(2 - \frac{n}{2} + int)|^2}{|\xi(\frac{n}{2} + int)|^2} \int_{SL_2(\mathbb{Z}) \backslash X_2} \phi(z) \left| E \left(z, 1 - \frac{n}{4} + int \right) \right|^2 d^*z \end{aligned}$$

We follow the proof of Proposition 2.1 of [LS] closely. Consider

$$\begin{aligned} I_\phi(s) &:= \int_{SL_2(\mathbb{Z}) \backslash X_2} \phi(z) E \left(z, 1 - \frac{n}{4} + int \right) E(z, s) d^*z \\ &= \int_0^\infty \int_0^1 \phi(z) E \left(z, 1 - \frac{n}{4} + int \right) y^s \frac{dx dy}{y^2}, \end{aligned}$$

we will substitute $s = 1 - \frac{n}{4} - int$ later in the proof. For odd forms this integral is 0, so we only consider even forms. Fourier expansions of even forms $\phi(z)$ and $E(z, s)$ are given by:

$$\phi(z) = y^{\frac{1}{2}} \sum_{n=1}^{\infty} \rho_\phi(1) \lambda_\phi(n) K_{it_\phi}(2\pi n y) \cos(2\pi n x),$$

$$E(z, s) = y^s + \frac{\xi(2s-1)}{\xi(2s)} y^{1-s} + \frac{2y^{\frac{1}{2}}}{\xi(2s)} \sum_{n=1}^{\infty} n^{s-\frac{1}{2}} \sigma_{1-2s}(n) K_{s-\frac{1}{2}}(2\pi n y) \cos(2\pi n x).$$

Since ϕ is fixed, we can ignore the normalization constant $\rho_\phi(1)$. The spectral parameter λ_ϕ is related to t_ϕ by $\lambda_\phi = \frac{1}{4} + t_\phi^2$. The L -function associated to u_ϕ is defined by

$$L(\phi, s) := \sum_{n=1}^{\infty} \frac{\lambda_\phi(n)}{n^s} = \prod_p (1 - \lambda_\phi(p) p^{-s} + p^{-2s})^{-1}.$$

So

$$\begin{aligned}
I_\phi(s) &= \int_0^\infty \int_0^1 \left(y^{\frac{1}{2}} \sum_{n=1}^\infty \lambda_\phi(n) K_{it_\phi}(2\pi ny) \cos(2\pi nx) \right) \left(y^{1-\frac{n}{4}+int} + \frac{\xi(1-\frac{n}{2}+2int)}{\xi(2-\frac{n}{2}+2int)} y^{\frac{n}{4}-int} \right. \\
&\quad \left. + \frac{2y^{\frac{1}{2}}}{\xi(2-\frac{n}{2}+2int)} \sum_{n'=1}^\infty n'^{\frac{1}{2}-\frac{n}{4}+int} \sigma_{\frac{n}{2}-1-2int}(n') K_{\frac{1}{2}-\frac{n}{4}+int}(2\pi n'y) \cos(2\pi n'x) \right) y^s \frac{dx dy}{y^2} \\
&= \frac{1}{\xi(2-\frac{n}{2}+2int)} \int_0^\infty \sum_{n=1}^\infty \lambda_\phi(n) K_{it_\phi}(2\pi ny) n^{\frac{1}{2}-\frac{n}{4}+int} \sigma_{\frac{n}{2}-1-2int}(n) K_{\frac{1}{2}-\frac{n}{4}+int}(2\pi ny) y^s \frac{dy}{y} \\
&= \frac{1}{\xi(2-\frac{n}{2}+2int)} \left(\sum_{n=1}^\infty \frac{\lambda_\phi(n) n^{\frac{1}{2}-\frac{n}{4}+int} \sigma_{\frac{n}{2}-1-2int}(n)}{n^s} \right) \int_0^\infty K_{it_\phi}(2\pi y) K_{\frac{1}{2}-\frac{n}{4}+int}(2\pi y) y^s \frac{dy}{y}.
\end{aligned}$$

Following equation (15) of [LS], it can be shown that

$$\sum_{n=1}^\infty \frac{\lambda_\phi(n) n^{\frac{1}{2}-\frac{n}{4}+int} \sigma_{\frac{n}{2}-1-2int}(n)}{n^s} = \frac{L(\phi, s - \frac{1}{2} + \frac{n}{4} - int) L(\phi, s + \frac{1}{2} - \frac{n}{4} + int)}{\zeta(2s)}.$$

The Mellin transform of product of K -Bessel functions can be evaluated using 4.1, we have

$$\begin{aligned}
&\int_0^\infty K_{it_\phi}(2\pi y) K_{\frac{1}{2}-\frac{n}{4}+int}(2\pi y) y^s \frac{dy}{y} \\
&= \frac{2^{-3}\pi^{-s}}{\Gamma(s)} \Gamma\left(\frac{s+it_\phi+\frac{1}{2}-\frac{n}{4}+int}{2}\right) \Gamma\left(\frac{s-it_\phi+\frac{1}{2}-\frac{n}{4}+int}{2}\right) \Gamma\left(\frac{s+it_\phi-\frac{1}{2}+\frac{n}{4}-int}{2}\right) \\
&\quad \times \Gamma\left(\frac{s-it_\phi-\frac{1}{2}+\frac{n}{4}-int}{2}\right).
\end{aligned}$$

Now let $s = 1 - \frac{n}{4} - int$, together we can estimate the following product by using Stirling's formula and the bound $L(u_j, \frac{1}{2} + it) \ll |t|^{\frac{1}{2}+\epsilon}$:

$$\begin{aligned}
&\frac{|\xi(2-\frac{n}{2}+int)|^2}{|\xi(\frac{n}{2}+int)|^2} \frac{L(\phi, \frac{1}{2}-2int) L(\phi, \frac{3-n}{2})}{\xi(2-\frac{n}{2}+2int) \xi(2-\frac{n}{2}-2int)} \Gamma\left(\frac{\frac{3-n}{2}+it_\phi}{2}\right) \Gamma\left(\frac{\frac{3-n}{2}-it_\phi}{2}\right) \\
&\quad \times \Gamma\left(\frac{\frac{1}{2}-2int+it_\phi}{2}\right) \Gamma\left(\frac{\frac{1}{2}-2int-it_\phi}{2}\right) \\
&\ll_{\phi, \epsilon} t^{-1} t^{\frac{1}{2}+\epsilon} = t^{-\frac{1}{2}+\epsilon}.
\end{aligned}$$

□

4.4. Cuspidal Contribution.

Proposition 4.2.

$$\int_{SL_n(\mathbb{Z}) \backslash X_n} \phi(z) E_{n-1,1}(z, s, 1) E_{n-1,1}(z, s', 1) d^*z = 0.$$

for $n \geq 3$ and ϕ a Maass cusp form on $GL(n)$.

Proof. Let $\phi(z)$ be a cusp form with Fourier-Whittaker expansion

$$\phi(z) = \sum_{\gamma \in U_{n-1}(\mathbb{Z}) \backslash SL_{n-1}(\mathbb{Z})} \sum_{m_1=1}^{\infty} \cdots \sum_{m_{n-2}=1}^{\infty} \sum_{m_{n-1} \neq 0} \frac{A(m_1, \dots, m_{n-1})}{\prod_{k=1}^{n-1} |m_k|^{k(n-k)/2}} W_{\text{Jac}}(My, \nu) e(m_1 x_1 + \cdots + m_{n-1} x_{n-1}) \Big|_{\gamma}$$

where

$$M = \begin{pmatrix} m_1 \cdots |m_n| & & & \\ & \ddots & & \\ & & m_1 & \\ & & & 1 \end{pmatrix}.$$

Note we define the slash operator to be such that

$$f(z)|_{\gamma} = f\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} \cdot z\right).$$

By the Rankin-Selberg convolution on $GL(n)$,

$$\begin{aligned} & \int_{SL_n(\mathbb{Z}) \backslash X_n} \phi(z) E_n(z, s', 1) E_n(z, s, 1) d^* z \\ &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_0^1 \cdots \int_0^1 \sum_{m_1=1}^{\infty} \cdots \sum_{m_{n-2}=1}^{\infty} \sum_{m_{n-1} \neq 0} \frac{A(m_1, \dots, m_{n-1})}{\prod_{k=1}^{n-1} |m_k|^{k(n-k)/2}} W_{\text{Jac}}(My, \nu) e(m_1 x_1 + \cdots + m_{n-1} x_{n-1}) \\ & \quad \times E_n(z, s', 1) \det(y)^s \prod_{1 \leq i < j \leq n-1} dx_{i,j} \prod_{k=1}^{n-1} \frac{dy_k}{y_k^{k(n-k)+1}}. \end{aligned}$$

To utilize the Fourier expansion

$$E_n(z, s, 1) = \sum_{m_1 \in \mathbb{Z}} \hat{\phi}_{(m_1, 0, \dots, 0)}(z) + \sum_{i=2}^{n-1} \sum_{\gamma_i \in P_i(\mathbb{Z}) \backslash SL_i(\mathbb{Z})} \sum_{m_i=1}^{\infty} \hat{\phi}_{(0, \dots, 0, m_i, 0, \dots, 0)} \left(\begin{pmatrix} \gamma_i & \\ & I_{n-i} \end{pmatrix} z \right),$$

we must understand the effect of $\begin{pmatrix} \gamma_i & \\ & I_{n-i} \end{pmatrix}$ on $x_{i,i+1}$. First we write

$$x = \begin{pmatrix} X_1 & V \\ & X_2 \end{pmatrix} \quad y = \begin{pmatrix} Y_1 y_1 \cdots y_{n-i-1} & \\ & Y_2 \end{pmatrix},$$

where $x \cdot y$ is the Iwasawa decomposition of z and X_1, Y_1 are of dimension $i \times i$, X_2, Y_2 are of dimension $(n-i) \times (n-i)$. Matrix multiplication gives

$$\begin{pmatrix} \gamma_i & \\ & I_{n-i} \end{pmatrix} \begin{pmatrix} X_1 & V \\ & X_2 \end{pmatrix} \begin{pmatrix} Y_1 y_1 \cdots y_{n-i-1} & \\ & Y_2 \end{pmatrix} = \begin{pmatrix} \gamma_i \cdot X_1 \cdot Y_1 & \gamma_i \cdot V \\ & X_2 \end{pmatrix} \begin{pmatrix} I_i y_1 \cdots y_{n-i-1} & \\ & Y_2 \end{pmatrix}.$$

By the Iwasawa decomposition, the matrix $\gamma_i \cdot X_1 \cdot Y_1$ can be written as $X'_1 \cdot Y'_1 \cdot K'_1$. So

$$\begin{pmatrix} \gamma_i & \\ & I_{n-i} \end{pmatrix} z = \begin{pmatrix} X'_1 & \gamma_i \cdot V \\ & X_2 \end{pmatrix} \begin{pmatrix} Y'_1 y_1 \cdots y_{n-i-1} & \\ & Y_2 \end{pmatrix} \begin{pmatrix} K' & \\ & 1 \end{pmatrix}.$$

We have put $\begin{pmatrix} \gamma_i & \\ & I_{n-i} \end{pmatrix} z$ in Iwasawa form. In particular the $(i, i+1)$ -entry becomes

$$\sum_{\ell=1}^i \gamma_{i,\ell} x_{\ell, i+1},$$

which appears in the bottom left corner of $\gamma_i \cdot V$.

It is important to notice that X'_1 and Y'_1 are completely determined by X_1, Y_1 and γ_i . They do not have any relations to the matrix V . Thus the exponential factor $e(m_i x_{i,i+1})$ in $\hat{\phi}_{(0,\dots,0,m_i,0,\dots,0)}(z)$ is the only place $x_{i,i+1}$ appears. As a result $\sum_{\ell=1}^i \gamma_{i,\ell} x_{\ell,i+1}$ only appears in the exponential after applying γ_i . Integrating against $x_{\ell,i+1}$ forces $\gamma_{\ell,i+1}$ to be 0 for $1 \leq \ell \leq i-1$. This also means that $\gamma_{i,i} = 1$ as $\gamma \in SL_{n-1}(\mathbb{Z})$. So only the identity coset in the sum over $P_i(\mathbb{Z}) \backslash SL_i(\mathbb{Z})$ contributes. Finally integrating against either $x_{i+1,i+2}$ or $x_{i-1,i}$ shows the cuspidal contribution is 0. This completes the proof. \square

4.5. Partition of type $n = n_1 + \dots + n_r$, $n_r \geq 2$. The goal of this section is to prove the following proposition.

Proposition 4.3. *Let $n = n_1 + \dots + n_r$ be a partition of n in non-increasing order with $n_r \geq 2$, we have*

$$\int_{SL_n(\mathbb{Z}) \backslash X_n} E_{n_1, \dots, n_r}(z, \eta, \phi_1, \dots, \phi_r) |E_{n-1,1}(z, s, 1)|^2 d^*z = 0.$$

Proof. By Lemma 4.5, the triple product can be written as

$$\begin{aligned} & \int_{SL_n(\mathbb{Z}) \backslash X_n} E_{n_1, \dots, n_r}(z, \phi_1, \dots, \phi_r, \eta) \overline{E_{n-1,1}(z, 1, s)} E_{n-1,1}(z, 1, s) d^*z \\ &= \int_{SL_{n-1}(\mathbb{Z}) \backslash X_{n-1}} \int_0^\infty \int_{(\mathbb{Z}/\mathbb{R})^{n-1}} E_{n_1, \dots, n_r}(z, \phi_1, \dots, \phi_r, \eta) \overline{E_{n-1,1}(z, 1, s)} \ell^{-ns} \left(-\frac{n}{n-1}\right) d\bar{x}_n \ell^n \frac{d\ell}{\ell} d^*z', \end{aligned}$$

where

$$\Xi(z') := \int_0^\infty \int_{(\mathbb{Z}/\mathbb{R})^{n-1}} E_{n_1, \dots, n_r}(\bar{x}_n, z', \ell, \phi_1, \dots, \phi_r, \eta) \overline{E_{n-1,1}(\bar{x}_n, z', \ell, 1, s)} \ell^{-ns} \left(-\frac{n}{n-1}\right) d\bar{x}_n \ell^n \frac{d\ell}{\ell}$$

is automorphic.

We have the Fourier expansion:

$$\begin{aligned} & E_{n_1, \dots, n_r}(z, \phi_1, \dots, \phi_r, \eta) \\ &= \sum_{m_1=0}^\infty \sum_{m_2=0}^\infty \sum'_{\gamma_2 \in P_{1,1} \backslash SL_2} \cdots \sum_{m_{n-1}=1}^\infty \sum_{\gamma_{n-1} \in P_{n-2,1} \backslash SL_{n-1}} a_{m_1, \dots, m_{n-1}} \\ & \quad \times W_{m_1, \dots, m_{n-1}}(Y) e(m_1 x_{1,2} + \dots + m_{n-1} x_{n-1,n}) \Big|_{\gamma_2 \cdots \gamma_{n-1}} \end{aligned}$$

By Lemma 2.1 we have that

- (1) $a_{m_1, \dots, m_{n-2}, 0} = 0$, hence, in the Fourier expansion above, the variable m_{n-1} starts at 1.
- (2) $a_{m_1, \dots, m_{n-4}, 0, 0, m_{n-1}} = 0$.
- (3) $a_{0, \dots, 0} = 0$.

We now use these properties to simplify $\Xi(z')$. Notice $m_{n-1} \neq 0$ in the expansion for $E_{n_1, \dots, n_r}(z, \phi_1, \dots, \phi_r, \eta)$, so $E_{n_1, \dots, n_r}(z, \phi_1, \dots, \phi_r, \eta)$ is orthogonal to all terms with $i < n-1$ in the Fourier expansion of $E_{(n-1,1)}(z, s, 1)$. Thus after executing the \bar{x}_n integrals we have

$$\begin{aligned} \Xi(z') &= \int_0^\infty \sum_{m_1=0}^\infty \sum_{m_2=0}^\infty \sum'_{\gamma_2 \in P_{1,1} \backslash SL_2} \cdots \sum_{m_{n-1}=1}^\infty \sum_{\gamma_{n-1} \in P_{n-2,1} \backslash SL_{n-1}} a_{m_1, \dots, m_{n-1}} b_{m_{n-1}} \\ (4.5) \quad & \times W_{m_1, \dots, m_{n-1}}(Y', \ell) e(m_1 x_{1,2} + \dots + m_{n-2} x_{n-2,n-1}) \Big|_{\gamma_2 \cdots \gamma_{n-1}} W'_{m_{n-1}}(Y', \ell) \Big|_{\gamma_{n-1}} \left(-\frac{n}{n-1}\right) \ell^n \frac{d\ell}{\ell}. \end{aligned}$$

Here $z' = X'Y'$ in Iwasawa form and we used the simplified notation for the Fourier coefficients of degenerate Eisenstein series:

$$\begin{aligned} b_{m_{n-1}} &= \frac{2}{\xi(ns)} |m_{n-1}|^{\frac{ns}{2}-\frac{1}{2}} \sigma_{-ns+1}(|m_{n-1}|) \\ W'(Y', \ell) &= K_{\frac{ns}{2}-\frac{1}{2}}(2\pi|m_{n-1}|y_1) (y_1^{n-1} y_2^{n-2} \cdots y_{n-1})^s y_1^{-\frac{ns}{2}+\frac{1}{2}} \\ &= K_{\frac{ns}{2}-\frac{1}{2}}(2\pi|m_{n-1}|y_1) \ell^{-ns} y_1^{-\frac{ns}{2}+\frac{1}{2}}. \end{aligned}$$

This corresponds to the term $\hat{\phi}_{(0,\dots,0,m_{n-1})}$ in Theorem 3.1. It is important to note that ℓ is invariant under the action of $\gamma_2, \dots, \gamma_{n-1}$, so we may move the integral inside all the summations:

$$\begin{aligned} \Xi(z') &= \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \sum'_{\gamma_2 \in P_{1,1} \backslash SL_2} \cdots \sum_{m_{n-1}=1}^{\infty} \sum_{\gamma_{n-1} \in P_{n-2,1} \backslash SL_{n-1}} a_{m_1, \dots, m_{n-1}} b_{m_{n-1}} \\ &\quad \times \int_0^{\infty} W_{m_1, \dots, m_{n-1}}(Y', \ell) e(m_1 x_{1,2} + \cdots m_{n-2} x_{n-2, n-1})|_{\gamma_2 \cdots \gamma_{n-1}} W'_{m_{n-1}}(Y', \ell)|_{\gamma_{n-1}} \left(-\frac{n}{n-1}\right) \ell^n \frac{d\ell}{\ell}. \end{aligned}$$

It's tempting to conclude here that $\Xi(z')$ is an automorphic function without constant term, so $\int_{SL_{n-1} \backslash X_{n-1}} \Xi(z') d^* z'$ must be 0. However this is not the correct form of Fourier expansion for $\Xi(z')$. Note that $\Xi(z')$ is a function on $SL_{n-1} \backslash X_{n-1}$. The correct form of Fourier expansion looks like

$$\sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \sum'_{\gamma_2 \in P_{1,1} \backslash SL_2} \cdots \sum_{m_{n-2}=0}^{\infty} \sum'_{\gamma_{n-2} \in P_{n-3,1} \backslash SL_{n-2}} a_{(m_1, \dots, m_{n-2})} f_{(m_1, \dots, m_{n-2})}(y) e(m_1 x_{1,2} + \cdots + m_{n-2} x_{n-2, n-1}).$$

In our expansion of $\Xi(z')$, we have a sum over $P_{n-2,1} \backslash SL_{n-1}$. But this allows us to unfold again.

$$\begin{aligned} &\int_{SL_{n-1} \backslash X_{n-1}} \Xi(z') d^* z' \\ &= \int_{P_{n-2,1} \backslash X_{n-1}} \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \sum'_{\gamma_2 \in P_{1,1} \backslash SL_2} \cdots \sum_{m_{n-2}=0}^{\infty} \sum'_{\gamma_{n-2} \in P_{n-3,1} \backslash SL_{n-2}} \sum_{m_{n-1}=1}^{\infty} a_{m_1, \dots, m_{n-1}} b_{m_{n-1}} \\ &\quad \times \int_0^{\infty} W_{m_1, \dots, m_{n-1}}(Y', \ell) e(m_1 x_{1,2} + \cdots m_{n-2} x_{n-2, n-1})|_{\gamma_2 \cdots \gamma_{n-2}} W'_{m_{n-1}}(Y', \ell) \left(-\frac{n}{n-1}\right) \ell^n \frac{d\ell}{\ell} d^* z' \end{aligned}$$

Now suppose $F(z)$ is an automorphic function on $SL_n(\mathbb{Z}) \backslash X_n$ in the form of $F(z) = \sum_{\gamma \in P_{n-1,1} \backslash SL_n(\mathbb{Z})} G(z)|_{\gamma}$. First of all, for this definition to make sense, we need $G(z)$ to be invariant under the group $P_{n-1,1}$. (This is clearly the case for $\Xi(z')$.) Then

$$\begin{aligned} &\int_{SL_n(\mathbb{Z}) \backslash X_n} F(z) d^* z \\ &= \int_{P_{n-1,1} \backslash X_n} G(z) d^* z \\ &= \int_{SL_{n-1}(\mathbb{Z}) \backslash X_{n-1}} \int_0^{\infty} \int_{(\mathbb{Z}/\mathbb{R})^{n-1}} G \left(\begin{pmatrix} I_{n-1} & \bar{x}_n \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} z' & \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} \ell^{-\frac{1}{n-1}} \cdot I_{n-1} & \\ & \ell \end{pmatrix} \right) \\ &\quad \times \left(-\frac{n}{n-1}\right) d\bar{x}_n \ell^n \frac{d\ell}{\ell} d^* z'. \end{aligned}$$

We must verify that

$$G'(z') := \int_0^{\infty} \int_{(\mathbb{Z}/\mathbb{R})^{n-1}} G \left(\begin{pmatrix} I_{n-1} & \bar{x}_n \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} z' & \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} \ell^{-\frac{1}{n-1}} \cdot I_{n-1} & \\ & \ell \end{pmatrix} \right) \left(-\frac{n}{n-1}\right) d\bar{x}_n \ell^n \frac{d\ell}{\ell}$$

is automorphic. Indeed,

$$\begin{aligned}
& G'(\gamma z') \\
&= \int_0^\infty \int_{(\mathbb{Z}/\mathbb{R})^{n-1}} W \left(\begin{pmatrix} I_{n-1} & \bar{x}_n \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} z' & \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} \ell^{-\frac{1}{n-1}} \cdot I_{n-1} & \\ & \ell \end{pmatrix} \right) \left(-\frac{n}{n-1} \right) d\bar{x}_n \ell^n \frac{d\ell}{\ell} \\
&= \int_0^\infty \int_{(\mathbb{Z}/\mathbb{R})^{n-1}} W \left(\begin{pmatrix} I_{n-1} & \gamma^{-1} \bar{x}_n \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} z' & \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} \ell^{-\frac{1}{n-1}} \cdot I_{n-1} & \\ & \ell \end{pmatrix} \right) \left(-\frac{n}{n-1} \right) d\bar{x}_n \ell^n \frac{d\ell}{\ell} \\
&= \int_0^\infty \int_{(\mathbb{Z}/\mathbb{R})^{n-1}} W \left(\begin{pmatrix} I_{n-1} & \bar{x}_n \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} z' & \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} \ell^{-\frac{1}{n-1}} \cdot I_{n-1} & \\ & \ell \end{pmatrix} \right) \left(-\frac{n}{n-1} \right) d\bar{x}_n \ell^n \frac{d\ell}{\ell} \\
&= G'(z').
\end{aligned}$$

So

$$\begin{aligned}
& \int_{SL_{n-1} \backslash X_{n-1}} \Xi(z') d^* z' \\
&= \int_{P_{n-2,1} \backslash X_{n-1}} \sum_{m_1=0}^\infty \sum_{m_2=0}^\infty \sum'_{\gamma_2 \in P_{1,1} \backslash SL_2} \cdots \sum_{m_{n-2}=0}^\infty \sum'_{\gamma_{n-2} \in P_{n-3,1} \backslash SL_{n-2}} \sum_{m_{n-1}=1}^\infty a_{m_1, \dots, m_{n-1}} b_{m_{n-1}} \\
&\quad \times \int_0^\infty W_{m_1, \dots, m_{n-1}}(Y', \ell) e(m_1 x_{1,2} + \cdots + m_{n-2} x_{n-2, n-1})|_{\gamma_2 \cdots \gamma_{n-2}} W'_{m_{n-1}}(Y', \ell) \left(-\frac{n}{n-1} \right) \ell^n \frac{d\ell}{\ell} d^* z' \\
&= \int_{SL_{n-2} \backslash X_{n-2}} \sum_{m_1=0}^\infty \sum_{m_2=0}^\infty \sum'_{\gamma_2 \in P_{1,1} \backslash SL_2} \cdots \sum_{m_{n-3}=1}^\infty \sum_{\gamma_{n-3} \in P_{n-4,1} \backslash SL_{n-3}} \sum_{m_{n-1}=1}^\infty a_{m_1, \dots, m_{n-3}, 0, m_{n-1}} b_{m_{n-1}} \\
&\quad \times \int_0^\infty \int_0^\infty W_{m_1, \dots, m_{n-3}, 0, m_{n-1}}(Y'', \ell', \ell) e(m_1 x_{1,2} + \cdots + m_{n-3} x_{n-3, n-2})|_{\gamma_2 \cdots \gamma_{n-3}} \\
&\quad \times W'_{m_{n-1}}(\ell', \ell) \left(-\frac{n}{n-1} \right) \ell^n \frac{d\ell}{\ell} \left(-\frac{n-1}{n-2} \right) \ell'^n \frac{d\ell'}{\ell'} d^* z'
\end{aligned}$$

with

$$z' = \begin{pmatrix} I_{n-2} & \bar{x}_{n-1} \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} z'' & \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} \ell'^{-\frac{1}{n-2}} \cdot I_{n-2} & \\ & \ell' \end{pmatrix}$$

and $\ell' = (y_2^{n-2} \cdots y_{n-1})^{-\frac{1}{n-1}}$.

Here we also note that $W'(Y', \ell)$ is really a function of y_1 and ℓ . But $y_1 = \ell^{-\frac{n}{n-1}} \ell'$ so $W'(Y', \ell)$ is a function of ℓ and ℓ' . And ℓ and ℓ' are invariant under the action of $\gamma_2, \dots, \gamma_{n-3}$.

Finally,

$$\begin{aligned}
& \sum_{m_1=0}^\infty \sum_{m_2=0}^\infty \sum'_{\gamma_2 \in P_{1,1} \backslash SL_2} \cdots \sum_{m_{n-3}=1}^\infty \sum_{\gamma_{n-3} \in P_{n-4,1} \backslash SL_{n-3}} \sum_{m_{n-1}=1}^\infty a_{m_1, \dots, m_{n-3}, 0, m_{n-1}} b_{m_{n-1}} \\
&\quad \times \int_0^\infty \int_0^\infty W_{m_1, \dots, m_{n-3}, 0, m_{n-1}}(Y'', \ell', \ell) W'_{m_{n-1}}(\ell', \ell) \left(-\frac{n}{n-1} \right) \ell^n \frac{d\ell}{\ell} \left(-\frac{n-1}{n-2} \right) \ell'^n \frac{d\ell'}{\ell'} \\
&\quad \times e(m_1 x_{1,2} + \cdots + m_{n-3} x_{n-3, n-2})|_{\gamma_2 \cdots \gamma_{n-3}}
\end{aligned}$$

is in proper Fourier expansion form and is automorphic without constant term. Thus the integral over $SL_{n-2} \backslash X_{n-2}$ is 0. \square

5. MAIN TERM FROM MINIMAL PARABOLIC CONTRIBUTION

The purpose of this section is to prove the following theorem.

Theorem 5.1.

$$\begin{aligned} & \int_{SL_n(\mathbb{Z}) \backslash X_n} E_{(1, \dots, 1)}(z, \eta) \left| E_{(n-1, 1)} \left(z, \frac{1}{2} + it, 1 \right) \right|^2 d^* z \\ &= \frac{2}{\xi(n)} \log(t) \left(\int_{SL_n(\mathbb{Z}) \backslash X_n} E_{(1, \dots, 1)}(z, \eta) d^* z \right) + O_{n, \eta}(1), \end{aligned}$$

as $t \rightarrow \infty$.

We begin the proof by evaluating the integral on the right hand side.

Proposition 5.1.

$$\int_{SL_n(\mathbb{Z}) \backslash X_n} E_{(1, \dots, 1)}(z, \eta) d^* z = \frac{1}{n^{n-2}} \tilde{\eta} \left(\frac{2}{n}, \dots, \frac{2}{n} \right).$$

Proof. Recall that the incomplete Eisenstein series associated to the minimal parabolic subgroup is given by

$$E_{(1, \dots, 1)}(z, \eta) = \sum_{\gamma \in P_{1, \dots, 1}(\mathbb{Z}) \backslash SL_n(\mathbb{Z})} \eta \left(\prod_{j=1}^{n-1} y_j^{b_{j,1}}, \prod_{j=1}^{n-1} y_j^{b_{j,2}}, \dots, \prod_{j=1}^{n-1} y_j^{b_{j,n-1}} \right) \Big|_{\gamma}$$

where

$$b_{j,k} = \begin{cases} jk & \text{if } j+k \leq n, \\ (n-j)(n-k) & \text{if } j+k \geq n. \end{cases}$$

We can unfold with respect to the sum over $P_{1, \dots, 1}(\mathbb{Z}) \backslash SL_n(\mathbb{Z})$:

$$\begin{aligned} & \int_{SL_n(\mathbb{Z}) \backslash X_n} E_{(1, \dots, 1)}(z, \eta) d^* z \\ &= \int_{P_{1, \dots, 1}(\mathbb{Z}) \backslash X_n} \eta \left(\prod_{j=1}^{n-1} y_j^{b_{j,1}}, \prod_{j=1}^{n-1} y_j^{b_{j,2}}, \dots, \prod_{j=1}^{n-1} y_j^{b_{j,n-1}} \right) d^* z \\ &= \int_{(\mathbb{R}^+)^{n-1}} \int_{N_{min}(\mathbb{Z}) \backslash N_{min}(\mathbb{R})} \eta \left(\prod_{j=1}^{n-1} y_j^{b_{j,1}}, \prod_{j=1}^{n-1} y_j^{b_{j,2}}, \dots, \prod_{j=1}^{n-1} y_j^{b_{j,n-1}} \right) \prod_{1 \leq \alpha < \beta \leq n} dx_{\alpha, \beta} \prod_{\ell=1}^{n-1} y_{\ell}^{-\ell(n-\ell)-1} dy_{\ell} \\ &= \int_{(\mathbb{R}^+)^{n-1}} \eta \left(\prod_{j=1}^{n-1} y_j^{b_{j,1}}, \prod_{j=1}^{n-1} y_j^{b_{j,2}}, \dots, \prod_{j=1}^{n-1} y_j^{b_{j,n-1}} \right) \prod_{\ell=1}^{n-1} y_{\ell}^{-\ell(n-\ell)-1} dy_{\ell}. \end{aligned}$$

At this moment, we make a change of variables:

$$w_k = \prod_{j=1}^{n-1} y_j^{b_{j,k}} \quad \text{for } 1 \leq k \leq n-1.$$

One can easily show that

$$y_j = \left(\frac{w_{n-j}^2}{w_{n-j-1} w_{n-j+1}} \right)^{\frac{1}{n}} \quad \text{for } 1 \leq j \leq n-1$$

where we set $w_0 = w_n = 1$. Then we can calculate the Jacobian of this change of variables

$$J = \frac{1}{n^{n-2}} (w_1 w_{n-1})^{-\frac{n-1}{n}} \prod_{k=2}^{n-2} w_k^{-1}.$$

At the same time,

$$\prod_{\ell=1}^{n-1} y_\ell^{-\ell(n-\ell)-1} = (w_1 w_{n-1})^{-\frac{3}{n}} \prod_{k=2}^{n-2} w_k^{-\frac{2}{n}}.$$

Thus

$$\begin{aligned} & \int_{SL_n(\mathbb{Z}) \backslash X_n} E_{(1, \dots, 1)}(z, \eta) d^* z \\ &= \int_{(\mathbb{R}^+)^{n-1}} \eta(w_1, \dots, w_{n-1}) \frac{1}{n^{n-2}} \prod_{k=1}^{n-1} w_k^{-\frac{2}{n}} \prod_{\ell=1}^{n-1} \frac{dw_\ell}{w_\ell} \\ &= \frac{1}{n^{n-2}} \tilde{\eta} \left(\frac{2}{n}, \dots, \frac{2}{n} \right) \end{aligned}$$

as desired. \square

To prove Theorem 5.1, we use the Fourier expansion of the degenerate Eisenstein series from Theorem 3.1. We cross multiply the Fourier expansion to open the square. There are four type of terms that we will evaluate separately:

(1)

$$\Delta_1(z, t) := \left| \hat{\phi}_{(0, \dots, 0)} \left(z, \frac{1}{2} + it \right) \right|^2;$$

(2)

$$\Delta_2(z, t) := \left| \sum_{m_1 \neq 0} \hat{\phi}_{(m_1, 0, \dots, 0)} \left(z, \frac{1}{2} + it \right) \right|^2;$$

(3) For $2 \leq k \leq n-1$

$$\Delta_{3,k}(z, t) := \left| \sum_{\gamma_i \in P_i(\mathbb{Z}) \backslash SL_k(\mathbb{Z})} \sum_{m_k=1}^{\infty} \hat{\phi}_{(0, \dots, 0, m_k, 0, \dots, 0)} \left(\begin{pmatrix} \gamma_i & \\ & I_{n-i} \end{pmatrix} z, \frac{1}{2} + it \right) \right|^2;$$

(4) All cross terms.

Lemma 5.1. *All cross terms vanish.*

Proof. A typical term after opening the square has the form

$$\sum_{\gamma_i \in P_i(\mathbb{Z}) \backslash SL_i(\mathbb{Z})} \sum_{m_i=1}^{\infty} \hat{\phi}_{(0, \dots, 0, m_i, 0, \dots, 0)} \left(\begin{pmatrix} \gamma_i & \\ & I_{n-i} \end{pmatrix} z \right) \sum_{\gamma_j \in P_j(\mathbb{Z}) \backslash SL_j(\mathbb{Z})} \sum_{m_j=1}^{\infty} \bar{\phi}_{(0, \dots, 0, m_j, 0, \dots, 0)} \left(\begin{pmatrix} \gamma_j & \\ & I_{n-j} \end{pmatrix} z \right).$$

If $i \neq j$, without loss of generality we can assume $i > j$. Then integrating against the variables $x_{1,k+1}, \dots, x_{k-1,k+1}$ forces γ_i to be the identity coset. Then the this cross term vanishes after integrating against $x_{k,k+1}$. \square

Lemma 5.2.

$$\int_{SL_n(\mathbb{Z}) \backslash X_n} E_{(1, \dots, 1)}(z, \eta) \Delta_1(z, t) d^* z = O(1).$$

Proof.

$$\Delta_1(z, t) = \left| \sum_{k=0}^{n-1} \frac{2\xi(k+1 - \frac{n}{2} + int)}{\xi(\frac{n}{2} + int)} \left(y_1 y_2^2 \cdots y_{n-k-1}^{n-k-1} \right)^{\frac{1}{2}-it} \left(y_{n-k}^k y_{n-(k-1)}^{k-1} \cdots y_{n-1} \right)^{\frac{1}{2}+it} \right|^2.$$

First we treat diagonal terms:

$$\begin{aligned} & \int_{SL_n(\mathbb{Z}) \backslash X_n} E_{(1, \dots, 1)}(z, \eta) \sum_{k=0}^{n-1} \frac{|\xi(k+1 - \frac{n}{2} + int)|^2}{|\xi(\frac{n}{2} + int)|^2} \left(y_1 y_2^2 \cdots y_{n-k-1}^{n-k-1} \right) \left(y_{n-k}^k y_{n-(k-1)}^{k-1} \cdots y_{n-1} \right) d^* z \\ &= \sum_{k=0}^{n-1} \frac{|\xi(k+1 - \frac{n}{2} + int)|^2}{|\xi(\frac{n}{2} + int)|^2} \int_0^\infty \cdots \int_0^\infty \eta \left(\prod_{j=1}^{n-1} y_j^{b_{j,1}}, \prod_{j=1}^{n-1} y_j^{b_{j,2}}, \dots, \prod_{j=1}^{n-1} y_j^{b_{j,n-1}} \right) \\ & \quad \times \left(y_1 y_2^2 \cdots y_{n-k-1}^{n-k-1} \right) \left(y_{n-k}^k y_{n-(k-1)}^{k-1} \cdots y_{n-1} \right) \prod_{\ell=1}^{n-1} y_\ell^{-\ell(n-\ell)-1} dy_\ell \\ &= \sum_{k=0}^{n-1} \frac{|\xi(k+1 - \frac{n}{2} + int)|^2}{|\xi(\frac{n}{2} + int)|^2} c_{k,\eta} \end{aligned}$$

for some constants $c_{k,\eta}$. We will not calculate these constants as they do not contribute to the main term. For $k=0, n-1$, we get a contribution of $O(1)$. By Stirling's formula the rest will contribute $O(t^{-1})$.

For off-diagonal terms, some powers of y will be imaginary unlike the diagonal terms. We can then use integration by parts repeatedly and get a contribution of $O(t^{-2016})$, say. \square

Lemma 5.3.

$$\begin{aligned} & \int_{SL_n(\mathbb{Z}) \backslash X_n} E_{(1, \dots, 1)}(z, \eta) \Delta_2(z, t) d^* z \\ &= \frac{2}{|\xi(\frac{n}{2} + int)|^2} \int_{(\mathbb{R}^+)^{n-2}} \frac{1}{(2\pi i)^{n-1}} \int_{(2)} \cdots \int_{(2)} \tilde{\eta}(\nu) \\ & \quad \times \frac{\xi\left(\sum_{j=1}^{n-1} b_{n-1,j} \nu_j\right) \xi\left(\sum_{j=1}^{n-1} b_{n-1,j} \nu_j - \frac{n}{2} + 1 + int\right)}{\xi\left(2\sum_{j=1}^{n-1} b_{n-1,j} \nu_j - n + 2\right)} \\ & \quad \times \xi\left(\sum_{j=1}^{n-1} b_{n-1,j} \nu_j - \frac{n}{2} + 1 - int\right) \xi\left(\sum_{j=1}^{n-1} b_{n-1,j} \nu_j - n + 2\right) \\ & \quad \times \left(y_1 y_2^2 \cdots y_{n-2}^{n-2} \right) \prod_{\ell=1}^{n-1} d\nu_\ell \prod_{k=1}^{n-2} \frac{dy_k}{y_k^{k(n-k)+1}} \end{aligned}$$

Proof.

$$\begin{aligned}
& \int_{SL_n(\mathbb{Z}) \backslash X_n} E_{(1, \dots, 1)}(z, \eta) \Delta_2(z, t) d^* z \\
&= \int_{SL_n(\mathbb{Z}) \backslash X_n} E_{(1, \dots, 1)}(z, \eta) \left| \sum_{m_1 \neq 0} \hat{\phi}_{(m_1, 0, \dots, 0)} \left(z, \frac{1}{2} + it \right) \right|^2 d^* z \\
&= \frac{8}{\left| \xi \left(\frac{n}{2} + int \right) \right|^2} \int_{(\mathbb{R}^+)^{n-1}} \int_{N_{\min(\mathbb{Z})} \backslash N_{\min(\mathbb{R})}} \frac{1}{(2\pi i)^{n-1}} \int_{(2)} \cdots \int_{(2)} \tilde{\eta}(\nu) \left(\prod_{i=1}^{n-1} y_i^{\sum_{j=1}^{n-1} b_{i,j} \nu_j} \right) \\
&\quad \times \sum_{m_1=1}^{\infty} m_1^{1-\frac{n}{2}} \sigma_{\frac{n}{2}-1-int}(m_1) \sigma_{\frac{n}{2}-1+int}(m_1) K_{\frac{1}{2}-\frac{n}{4}+\frac{int}{2}}(2\pi m_1 y_{n-1}) K_{\frac{1}{2}-\frac{n}{4}-\frac{int}{2}}(2\pi m_1 y_{n-1}) \\
&\quad \times (y_1 y_2^2 \cdots y_{n-2}^{n-2}) y_{n-1}^{\frac{n}{2}} \prod_{\ell=1}^{n-1} d\nu_{\ell} \prod_{k=1}^{n-1} \frac{dy_k}{y_k^{k(n-k)+1}}
\end{aligned}$$

The Lemma is then proved by applying Lemma 4.2 to convert the divisor sum into Zeta functions and 4.1 to convert K-Bessel integrals to Gamma functions. \square

Lemma 5.4.

$$\begin{aligned}
& \int_{SL_n(\mathbb{Z}) \backslash X_n} E_{(1, \dots, 1)}(z, \eta) \Delta_{3,k}(z, t) d^* z \\
&= \frac{2}{\left| \xi \left(\frac{n}{2} + int \right) \right|^2} \int_{(\mathbb{R}^+)^{n-2}} \frac{1}{(2\pi i)^{n-1}} \int_{(2)} \cdots \int_{(2)} \tilde{\eta}(\nu) \\
&\quad \times \sum_{h=0}^{k-1} \frac{\xi \left((1-k)(n-k) + \sum_{j=1}^{n-1} b_{n-k,j} \nu_j \right) \xi \left((1-k)(n-k) + \sum_{j=1}^{n-1} b_{n-k,j} \nu_j - \frac{n}{2} + k + int \right)}{\xi \left(2(1-k)(n-k) + 2 \sum_{j=1}^{n-1} b_{n-k,j} \nu_j - n + 2k \right)} \\
&\quad \times \xi \left((1-k)(n-k) + \sum_{j=1}^{n-1} b_{n-k,j} \nu_j - \frac{n}{2} + k - int \right) \xi \left(\sum_{j=1}^{n-1} b_{n-k,j} \nu_j - k(n-k) + h + 1 \right) \\
&\quad \times \prod_{i=1}^{n-k-1} y_i^{\sum_{j=1}^{n-1} b_{i,j} \nu_j + i - i(n-i)} \prod_{i=n-k+1}^{n-h-1} y_i^{k(n-k) - i(n-i) + \sum_{j=1}^{n-1} (b_{i,j} - b_{n-k,j}) \nu_j} \prod_{i=n-h}^{n-1} y_i^{\sum_{j=1}^{n-1} b_{i,j} \nu_j + n - i - i(n-i)} \\
&\quad \prod_{\ell=1}^{n-1} d\nu_{\ell} \prod_{\substack{i=1 \\ i \neq n-k}}^{n-1} \frac{dy_i}{y_i}
\end{aligned}$$

Proof. By Lemma 3.3, as γ_i maps $x_{i,i+1}$ to $a_1 x_{1,i+1} + a_2 x_{2,i+1} + \cdots + a_i x_{i,i+1}$ where the primitive vector (a_1, \dots, a_i) is the bottom row of γ_i . Now integrating the exponential factor

$$e \left(m_i (a_1 x_{1,i+1} + a_2 x_{2,i+1} + \cdots + a_i x_{i,i+1}) - m'_i (a'_1 x_{1,i+1} + a'_2 x_{2,i+1} + \cdots + a'_i x_{i,i+1}) \right)$$

in the diagonal term forces $m_i = m'_i$ and $\gamma_i = \gamma'_i$ up to a sign. Thus the k -th ($2 \leq k \leq n-1$) diagonal term Ξ_k reads with $s = \frac{1}{2} + it$:

$$\begin{aligned}
\Xi_k &:= \frac{8}{\xi \left(\frac{n}{2} + int \right) \xi \left(\frac{n}{2} - int \right)} \sum_{\gamma_k \in P_k(\mathbb{Z}) \backslash SL_k(\mathbb{Z})} \sum_{m_k=1}^{\infty} |m_k|^{k-\frac{n}{2}} \sigma_{\frac{n}{2}-k-int}(|m_k|) \sigma_{\frac{n}{2}-k+int}(|m_k|) \\
&\quad \times K_{\frac{k}{2}-\frac{n}{4}+\frac{int}{2}}(2\pi |m_k| y'_{n-k}) K_{\frac{k}{2}-\frac{n}{4}-\frac{int}{2}}(2\pi |m_k| y'_{n-k}) \left(y_1 y_2^2 \cdots y_{n-k-1}^{n-k-1} \right) \left(y_{n-k}^k y_{n-(k-1)}^{k-1} \cdots y_{n-1} \right) y_{n-k}^{\frac{n}{2}-k}
\end{aligned}$$

Putting all terms together, we get

$$\Omega_k := \int_{(\mathbb{R}^+)^{n-1}} \int_{N_{\min(\mathbb{Z})} \setminus N_{\min(\mathbb{R})}} \frac{1}{(2\pi i)^{n-1}} \int_{(2)} \cdots \int_{(2)} \prod_{i=1}^{n-1} y_i^{\sum_{j=1}^{n-1} b_{i,j} \nu_j} \Xi_k \prod_{\ell=1}^{n-1} d\nu_\ell \prod_{1 \leq i < j \leq n-1} dx_{i,j} \prod_{k=1}^{n-1} \frac{dy_k}{y_k^{k(n-k)+1}}.$$

We now make the change of variable $|m_k| y'_{n-k} \mapsto y_{n-k}$ and let $\Theta_k = (b_1^2 + b_2^2 + \cdots + b_k^2)^{\frac{1}{2}}$. So we have

$$\begin{aligned} \Omega_k &= \frac{8}{\xi\left(\frac{n}{2} + int\right) \xi\left(\frac{n}{2} - int\right)} \int_{(\mathbb{R}^+)^{n-1}} \int_{N_{\min(\mathbb{Z})} \setminus N_{\min(\mathbb{R})}} \frac{1}{(2\pi i)^{n-1}} \int_{(2)} \cdots \int_{(2)} \tilde{\eta}(\nu) \left(\prod_{i=1}^{n-1} y_i^{\sum_{j=1}^{n-1} b_{i,j} \nu_j} \right) \\ &\times \sum_{\gamma_k \in P_k(\mathbb{Z}) \setminus SL_k(\mathbb{Z})} \sum_{m_k=1}^{\infty} |m_k|^{k-\frac{n}{2}} \sigma_{\frac{n}{2}-k-int}(|m_k|) \sigma_{\frac{n}{2}-k+int}(|m_k|) (|m_k| \Theta_k)^{-\sum_{j=1}^{n-1} b_{n-k,j} \nu_j - k + k(n-k)} |m_k|^{k-\frac{n}{2}} \\ &\times K_{\frac{k}{2}-\frac{n}{4}+\frac{int}{2}}(2\pi y_{n-k}) K_{\frac{k}{2}-\frac{n}{4}-\frac{int}{2}}(2\pi y_{n-k}) \left(y_1 y_2^2 \cdots y_{n-k-1}^{n-k-1} \right) \left(y_{n-k}^k y_{n-(k-1)}^{k-1} \cdots y_{n-1} \right) y_{n-k}^{\frac{n}{2}-k} \\ &\prod_{\ell=1}^{n-1} d\nu_\ell \prod_{1 \leq i < j \leq n-1} dx_{i,j} \prod_{k=1}^{n-1} \frac{dy_k}{y_k^{k(n-k)+1}}. \end{aligned}$$

We use the identity

$$\sum_{n=1}^{\infty} \frac{\sigma_a(n) \sigma_b(n)}{n^s} = \frac{\zeta(s) \zeta(s-a) \zeta(s-b) \zeta(s-a-b)}{\zeta(2s-a-b)}.$$

to convert the sum over m_k to ζ functions:

$$\begin{aligned} &\sum_{m_k=1}^{\infty} \frac{\sigma_{\frac{n}{2}-k-int}(|m_k|) \sigma_{\frac{n}{2}-k+int}(|m_k|)}{|m_k|^{n-k-k(n-k)+\sum_{j=1}^{n-1} b_{n-k,j} \nu_j}} \\ &= \frac{\zeta\left((1-k)(n-k) + \sum_{j=1}^{n-1} b_{n-k,j} \nu_j\right) \zeta\left((1-k)(n-k) + \sum_{j=1}^{n-1} b_{n-k,j} \nu_j - \frac{n}{2} + k + int\right)}{\zeta\left(2(1-k)(n-k) + 2\sum_{j=1}^{n-1} b_{n-k,j} \nu_j - n + 2k\right)} \\ &\times \zeta\left((1-k)(n-k) + \sum_{j=1}^{n-1} b_{n-k,j} \nu_j - \frac{n}{2} + k - int\right) \zeta\left((1-k)(n-k) + \sum_{j=1}^{n-1} b_{n-k,j} \nu_j - n + 2k\right). \end{aligned}$$

Now we notice that

$$\sum_{\gamma_k \in P_k(\mathbb{Z}) \setminus SL_k(\mathbb{Z})} \Theta_k^{-\sum_{j=1}^{n-1} b_{n-k,j} \nu_j - k + k(n-k)}$$

is just a degenerate Eisenstein series. Integrating over x simply gives the constant term. It reads

$$\sum_{h=0}^{k-1} \frac{2\xi\left(\sum_{j=1}^{n-1} b_{n-k,j} \nu_j - k(n-k) + h + 1\right) (y_{(n-k)+1} \cdots y_{(n-k)+(k-h-1)+})^{-\sum_{j=1}^{n-1} b_{n-k,j} \nu_j + k(n-k) - h - 1}}{\xi\left(\sum_{j=1}^{n-1} b_{n-k,j} \nu_j - k(n-k) + k\right) \left(y_{(n-k)+1}^{k-h-2} \cdots y_{(n-k)+(k-h-2)+}\right)}$$

Now we use the identity

$$\int_0^\infty K_\mu(y) K_\nu(y) y^s \frac{dy}{y} = 2^{s-3} \frac{\Gamma\left(\frac{s+\mu+\nu}{2}\right) \Gamma\left(\frac{s+\mu-\nu}{2}\right) \Gamma\left(\frac{s-\mu+\nu}{2}\right) \Gamma\left(\frac{s-\mu-\nu}{2}\right)}{\Gamma(s)}$$

to evaluate the integral of product K -Bessel functions:

$$\int_0^\infty K_{\frac{k}{2}-\frac{n}{4}+\frac{int}{2}}(2\pi y_{n-k}) K_{\frac{k}{2}-\frac{n}{4}-\frac{int}{2}}(2\pi y_{n-k}) y_{n-k}^{\sum_{j=1}^{n-1} b_{n-k,j} \nu_j + \frac{n}{2} - k(n-k)} \frac{dy_{n-k}}{y_{n-k}}.$$

These γ factors will form completed ζ -functions:

$$\frac{1}{8} \frac{\xi\left((1-k)(n-k) + \sum_{j=1}^{n-1} b_{n-k,j} \nu_j\right) \xi\left((1-k)(n-k) + \sum_{j=1}^{n-1} b_{n-k,j} \nu_j - \frac{n}{2} + k + int\right)}{\xi\left(2(1-k)(n-k) + 2\sum_{j=1}^{n-1} b_{n-k,j} \nu_j - n + 2k\right)} \\ \times \xi\left((1-k)(n-k) + \sum_{j=1}^{n-1} b_{n-k,j} \nu_j - \frac{n}{2} + k - int\right) \xi\left((1-k)(n-k) + \sum_{j=1}^{n-1} b_{n-k,j} \nu_j - n + 2k\right).$$

Now we gather y_i factors for $i \neq n-k$:

$$\frac{\left(\prod_{\substack{i=1 \\ i \neq n-k}}^{n-1} y_i^{\sum_{j=1}^{n-1} b_{i,j} \nu_j}\right) \left(y_1 y_2^2 \cdots y_{n-k-1}^{n-k-1}\right) \left(y_{n-(k-1)}^{k-1} \cdots y_{n-1}\right)}{\left(y_{(n-k)+1}^{k-h-2} \cdots y_{(n-k)+(k-h-2)+}\right) \prod_{\substack{\ell=1 \\ \ell \neq n-k}}^{n-1} y_\ell^{\ell(n-\ell)}} \\ \times \left(y_{(n-k)+1} \cdots y_{(n-k)+(k-h-1)+}\right)^{-\sum_{j=1}^{n-1} b_{n-k,j} \nu_j + k(n-k) - h - 1} \\ = \prod_{i=1}^{n-k-1} y_i^{\sum_{j=1}^{n-1} b_{i,j} \nu_j + i - i(n-i)} \prod_{i=n-k+1}^{n-h-1} y_i^{k(n-k) - i(n-i) + \sum_{j=1}^{n-1} (b_{i,j} - b_{n-k,j}) \nu_j} \prod_{i=n-h}^{n-1} y_i^{\sum_{j=1}^{n-1} b_{i,j} \nu_j + n - i - i(n-i)}.$$

Now we put everything together,

$$\Omega_k = \frac{2}{|\xi(\frac{n}{2} + int)|^2} \int_{(\mathbb{R}^+)^{n-2}} \frac{1}{(2\pi i)^{n-1}} \int_{(2)} \cdots \int_{(2)} \tilde{\eta}(\nu) \\ \times \sum_{h=0}^{k-1} \frac{\xi\left((1-k)(n-k) + \sum_{j=1}^{n-1} b_{n-k,j} \nu_j\right) \xi\left((1-k)(n-k) + \sum_{j=1}^{n-1} b_{n-k,j} \nu_j - \frac{n}{2} + k + int\right)}{\xi\left(2(1-k)(n-k) + 2\sum_{j=1}^{n-1} b_{n-k,j} \nu_j - n + 2k\right)} \\ \times \xi\left((1-k)(n-k) + \sum_{j=1}^{n-1} b_{n-k,j} \nu_j - \frac{n}{2} + k - int\right) \xi\left(\sum_{j=1}^{n-1} b_{n-k,j} \nu_j - k(n-k) + h + 1\right) \\ \times \prod_{i=1}^{n-k-1} y_i^{\sum_{j=1}^{n-1} b_{i,j} \nu_j + i - i(n-i)} \prod_{i=n-k+1}^{n-h-1} y_i^{k(n-k) - i(n-i) + \sum_{j=1}^{n-1} (b_{i,j} - b_{n-k,j}) \nu_j} \prod_{i=n-h}^{n-1} y_i^{\sum_{j=1}^{n-1} b_{i,j} \nu_j + n - i - i(n-i)} \\ \prod_{\ell=1}^{n-1} d\nu_\ell \prod_{\substack{i=1 \\ i \neq n-k}}^{n-1} \frac{dy_i}{y_i}$$

□

Combining the previous two lemmas, we let

$$\begin{aligned}
\Omega_{k,h} &:= \frac{2}{|\xi(\frac{n}{2} + int)|^2} \int_{(\mathbb{R}^+)^{n-2}} \frac{1}{(2\pi i)^{n-1}} \int_{(2)} \cdots \int_{(2)} \tilde{\eta}(\nu) \\
&\quad \times \frac{\xi\left((1-k)(n-k) + \sum_{j=1}^{n-1} b_{n-k,j} \nu_j\right) \xi\left((1-k)(n-k) + \sum_{j=1}^{n-1} b_{n-k,j} \nu_j - \frac{n}{2} + k + int\right)}{\xi\left(2(1-k)(n-k) + 2\sum_{j=1}^{n-1} b_{n-k,j} \nu_j - n + 2k\right)} \\
&\quad \times \xi\left((1-k)(n-k) + \sum_{j=1}^{n-1} b_{n-k,j} \nu_j - \frac{n}{2} + k - int\right) \xi\left(\sum_{j=1}^{n-1} b_{n-k,j} \nu_j - k(n-k) + h + 1\right) \\
&\quad \times \prod_{i=1}^{n-k-1} y_i^{\sum_{j=1}^{n-1} b_{i,j} \nu_j + i - i(n-i)} \prod_{i=n-k+1}^{n-h-1} y_i^{k(n-k) - i(n-i) + \sum_{j=1}^{n-1} (b_{i,j} - b_{n-k,j}) \nu_j} \prod_{i=n-h}^{n-1} y_i^{\sum_{j=1}^{n-1} b_{i,j} \nu_j + n - i - i(n-i)} \\
&\quad \prod_{\ell=1}^{n-1} d\nu_\ell \prod_{\substack{i=1 \\ i \neq n-k}}^{n-1} \frac{dy_i}{y_i}
\end{aligned}$$

for $1 \leq k \leq n-1$ and $0 \leq h \leq k-1$, then we claim that the main term comes from $\Omega_{n-1,0}$. We have

$$\begin{aligned}
\Omega_{n-1,0} &= \frac{2}{|\xi(\frac{n}{2} + int)|^2} \int_{(\mathbb{R}^+)^{n-2}} \frac{1}{(2\pi i)^{n-1}} \int_{(2)} \cdots \int_{(2)} \tilde{\eta}(\nu) \\
&\quad \times \frac{\xi\left(2-n + \sum_{j=1}^{n-1} b_{1,j} \nu_j\right)^2 \xi\left(1 - \frac{n}{2} + \sum_{j=1}^{n-1} b_{n-k,j} \nu_j + int\right) \xi\left(1 - \frac{n}{2} + \sum_{j=1}^{n-1} b_{n-k,j} \nu_j - int\right)}{\xi\left(2-n + 2\sum_{j=1}^{n-1} b_{n-k,j} \nu_j\right)} \\
&\quad \times \prod_{i=2}^{n-1} y_i^{n-1-i(n-i) + \sum_{j=1}^{n-1} (b_{i,j} - b_{1,j}) \nu_j} \prod_{\ell=1}^{n-1} d\nu_\ell \prod_{\substack{i=1 \\ i \neq n-k}}^{n-1} \frac{dy_i}{y_i}
\end{aligned}$$

Lemma 5.5.

$$\Omega_{n-1,0} = \frac{2\tilde{\eta}\left(\frac{2}{n}, \dots, \frac{2}{n}\right) \log t}{\xi(n)n^{n-2}} + O(1).$$

Proof. Notice that there are $n-2$ Mellin transforms and $n-1$ Mellin inversion in the above expression. Thus we can take advantage of the Mellin inversion theorem. We make the change of variables

$$(5.1) \quad \sum_{j=1}^{n-1} (b_{i,j} - b_{1,j}) \nu_j \mapsto s_{i-1} \quad \text{for } 2 \leq i \leq n-1$$

and $s_{n-1} = v_{n-1}$. The Jacobian is calculated to be $\frac{1}{n^{n-3}}$ and explicitly we have

$$\begin{aligned}
v_1 &= \frac{s_1 - s_{n-3} + 2s_{n-2}}{n} + v_{n-1}, \\
v_i &= \frac{2s_{n-i-1} - s_{n-i} - s_{n-i-2}}{n} \quad \text{for } 2 \leq i \leq n-3, \\
v_{n-2} &= \frac{2s_1 - s_2}{8} \\
v_{n-1} &= s_{n-1}.
\end{aligned}$$

By 5.1, we also have $\sum_{j=1}^{n-1} b_{1,j} \nu_j \mapsto s_1 + ns_{n-1}$. So

$$\begin{aligned}
\Omega_{n-1,0} &= \frac{1}{n^{n-3}} \frac{2}{\left| \xi \left(\frac{n}{2} + int \right) \right|^2} \int_{(\mathbb{R}^+)^{n-2}} \frac{1}{(2\pi i)^{n-1}} \int_{(\sigma_1)} \cdots \int_{(\sigma_{n-1})} \\
&\quad \tilde{\eta} \left(\frac{s_1 - s_{n-3} + 2s_{n-2}}{n} + s_{n-1}, \dots, \frac{2s_{n-i-1} - s_{n-i} - s_{n-i-2}}{n}, \dots, \frac{2s_1 - s_2}{n}, s_{n-1} \right) \\
&\quad \times \frac{\xi(2 - n + s_1 + ns_{n-1})^2 \xi \left(1 - \frac{n}{2} + s_1 + ns_{n-1} + int \right) \xi \left(1 - \frac{n}{2} + s_1 + ns_{n-1} - int \right)}{\xi(2 - n + 2s_1 + 2ns_{n-1})} \\
&\quad \times \prod_{i=2}^{n-1} y_i^{n-1-i(n-i)+s_{i-1}} \prod_{\ell=1}^{n-1} ds_\ell \prod_{\substack{i=1 \\ i \neq n-k}}^{n-1} \frac{dy_i}{y_i} \\
&= \frac{1}{n^{n-3}} \frac{2}{\left| \xi \left(\frac{n}{2} + int \right) \right|^2} \frac{1}{2\pi i} \int_{(\sigma_{n-1})} \tilde{\eta} \left(s_{n-1}, \frac{2}{n}, \dots, \frac{2}{n}, s_{n-1} \right) \\
&\quad \times \frac{\xi(ns_{n-1} - 1)^2 \xi \left(\frac{n}{2} - 2 + ns_{n-1} + int \right) \xi \left(\frac{n}{2} - 2 + ns_{n-1} - int \right)}{\xi(n - 4 + 2ns_{n-1})} ds_{n-1}
\end{aligned}$$

We now shift contour from σ_{n-1} to $\frac{2-\Delta}{n}$ for some small $\Delta > 0$. In the process, we pick up the residue corresponding to the pole of the $\zeta(ns_{n-1} - 1)^2$ term at $s_{n-1} = \frac{2}{n}$. Hence

$$\Omega_{n-1,0} = R_1 + R_2$$

where

$$\begin{aligned}
R_1 &:= \frac{2\tilde{\eta} \left(\frac{2}{n}, \dots, \frac{2}{n} \right)}{\xi(n)n^{n-1}} \left(O(1) + \frac{n}{2} \left(\frac{\Gamma'}{\Gamma} \left(\frac{n}{4} + \frac{int}{2} \right) + \frac{\Gamma'}{\Gamma} \left(\frac{n}{4} - \frac{int}{2} \right) \right) \right) \\
&= \frac{2\tilde{\eta} \left(\frac{2}{n}, \dots, \frac{2}{n} \right) \log t}{\xi(n)n^{n-2}} + O(1),
\end{aligned}$$

$$\begin{aligned}
R_2 &:= \frac{1}{n^{n-3}} \frac{2}{\left| \xi \left(\frac{n}{2} + int \right) \right|^2} \frac{1}{2\pi i} \int_{\left(\frac{2-\Delta}{n} \right)} \tilde{\eta} \left(s_{n-1}, \frac{2}{n}, \dots, \frac{2}{n}, s_{n-1} \right) \\
&\quad \times \frac{\xi(ns_{n-1} - 1)^2 \xi \left(\frac{n}{2} - 2 + ns_{n-1} + int \right) \xi \left(\frac{n}{2} - 2 + ns_{n-1} - int \right)}{\xi(n - 4 + 2ns_{n-1})} ds_{n-1}.
\end{aligned}$$

Let $f(t_2) = \tilde{\eta} \left(\frac{2-\Delta}{n} + it_2, \frac{2}{n}, \dots, \frac{2}{n}, \frac{2-\Delta}{n} + it_2 \right)$, then $f(t_s)$ is a function with rapid decay in t_2 . We can bound the R_2 integral as follows:

$$\begin{aligned}
R_2 &\ll \int_{-\infty}^{\infty} |f(t_2)| \frac{|\zeta(1 - \Delta + int_2)|^2 |\Gamma \left(\frac{1-\Delta}{2} + \frac{int_2}{2} \right)|^2}{\zeta(n - 2\Delta + 2int_2) \Gamma \left(\frac{n}{2} - \Delta + int_2 \right)} \\
&\quad \times \frac{\left| \zeta \left(\frac{n}{2} - \Delta + in(t_2 + t) \right) \right| \left| \zeta \left(\frac{n}{2} - \Delta + in(t_2 - t) \right) \right| \left| \Gamma \left(\frac{n}{4} - \frac{\Delta}{2} + \frac{in(t_2+t)}{2} \right) \right| \left| \Gamma \left(\frac{n}{4} - \frac{\Delta}{2} + \frac{in(t_2-t)}{2} \right) \right|}{\left| \zeta \left(\frac{n}{2} + int \right) \right|^2 \left| \Gamma \left(\frac{n}{4} + \frac{int}{2} \right) \right|^2} dt_2
\end{aligned}$$

By Stirling's formula,

$$\begin{aligned}
& \frac{\left| \Gamma\left(\frac{n}{4} - \frac{\Delta}{2} + \frac{in(t_2+t)}{2}\right) \right| \left| \Gamma\left(\frac{n}{4} - \frac{\Delta}{2} + \frac{in(t_2-t)}{2}\right) \right|}{\left| \Gamma\left(\frac{n}{4} + \frac{int}{2}\right) \right|^2} \\
& \ll \frac{e^{-\frac{\pi n}{4}(|t_2+t|+|t_2-t|)} |t_2+t|^{\frac{n}{4}-\frac{1}{2}-\frac{\Delta}{2}} |t_2-t|^{\frac{n}{4}-\frac{1}{2}-\frac{\Delta}{2}}}{e^{-\frac{\pi nt}{2}} |t|^{\frac{n}{2}-1}} \\
& \ll \frac{|t_2+t|^{\frac{n}{4}-\frac{1}{2}-\frac{\Delta}{2}} |t_2-t|^{\frac{n}{4}-\frac{1}{2}-\frac{\Delta}{2}}}{|t|^{\frac{n}{2}-1}}.
\end{aligned}$$

So

$$R_2 \ll t^{-\Delta}.$$

□

Now we tackle $\Omega_{k,h}$ for general k, h .

Proposition 5.2. *For $(k, h) \neq (n-1, 0)$, we have*

$$\Omega_{k,h} = O(1).$$

Proof. We make the convenient change of variable

$$\begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_{n-1} \end{pmatrix} = M_{k,h} \cdot (b_{i,j}) \cdot \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \end{pmatrix}$$

where the matrix $M_{k,h}$ is the identity matrix except for the entries $(j, n-k)$ being -1 for $n-k+1 \leq j \leq n-h-1$. With this change variable we have

$$\begin{aligned}
\Omega_{k,h} &= \frac{2}{n^{n-2} \left| \xi\left(\frac{n}{2} + int\right) \right|^2} \int_{(\mathbb{R}^+)^{n-2}} \frac{1}{(2\pi i)^{n-1}} \int_{(\sigma_1)} \cdots \int_{(\sigma_{n-1})} \tilde{\eta}(v_1, \dots, v_{n-1}) \\
&\quad \times \frac{\xi((1-k)(n-k) + s_{n-k}) \xi(s_{n-k} - k(n-k) + h+1)}{\xi(2(1-k)(n-k) + 2s_{n-k} - n + 2k)} \\
&\quad \times \xi\left((1-k)(n-k) + s_{n-k} - \frac{n}{2} + k - int\right) \xi\left((1-k)(n-k) + s_{n-k} - \frac{n}{2} + k + int\right) \\
&\quad \times \prod_{i=1}^{n-k-1} y_i^{s_i+i-i(n-i)} \prod_{i=n-k+1}^{n-h-1} y_i^{k(n-k)-i(n-i)+s_i} \prod_{i=n-h}^{n-1} y_i^{s_i+n-i-i(n-i)} \prod_{\ell=1}^{n-1} ds_\ell \prod_{\substack{i=1 \\ i \neq n-k}}^{n-1} \frac{dy_i}{y_i} \\
&= \frac{2}{n^{n-2} \left| \xi\left(\frac{n}{2} + int\right) \right|^2} \frac{1}{2\pi i} \int_{(\sigma_{n-1})} f(s_{n-1}) \frac{\xi((1-k)(n-k) + s_{n-k}) \xi(s_{n-k} - k(n-k) + h+1)}{\xi(2(1-k)(n-k) + 2s_{n-k} - n + 2k)} \\
&\quad \times \xi\left((1-k)(n-k) + s_{n-k} - \frac{n}{2} + k - int\right) \xi\left((1-k)(n-k) + s_{n-k} - \frac{n}{2} + k + int\right) ds_{n-1}.
\end{aligned}$$

Now we will shift contours to $\text{Re}(s_{n-k}) = 1 - (1-k)(n-k) - \Delta$ for small $\Delta > 0$. There are two cases to consider.

Case 1: $n-k = h+1$ and $k \neq n-1$.

In this situation, $\xi((1-k)(n-k) + s_{n-k}) = \xi(s_{n-k} - k(n-k) + h+1)$. As a result, we will encounter a pole of order 2 during the contour shift at $s_{n-1} = 1 - (1-k)(n-k)$. So

$$\Omega_{k,h} = R_1 + R_2$$

where

$$R_1 = \frac{C_1 \xi \left(k + 1 - \frac{n}{2}\right) \xi \left(k + 1 - \frac{n}{2}\right)}{\left|\xi \left(\frac{n}{2} + int\right)\right|^2} \left(C_2 + \frac{\xi'}{\xi} \left(1 + k - \frac{n}{2} + int\right) + \frac{\xi'}{\xi} \left(1 + k - \frac{n}{2} - int\right) \right)$$

and

$$\begin{aligned} R_2 &= \frac{2}{n^{n-2} \left|\xi \left(\frac{n}{2} + int\right)\right|^2} \frac{1}{2\pi i} \int_{(1-(1-k)(n-k)-\Delta)} f(s_{n-1}) \frac{\xi((1-k)(n-k) + s_{n-k})^2}{\xi(2(1-k)(n-k) + 2s_{n-k} - n + 2k)} \\ &\quad \times \xi \left((1-k)(n-k) + s_{n-k} - \frac{n}{2} + k - int \right) \xi \left((1-k)(n-k) + s_{n-k} - \frac{n}{2} + k + int \right) ds_{n-1}. \end{aligned}$$

By Stirling's formula, one can show that

$$R_1 + R_2 \ll t^{-\frac{1}{2}}.$$

Case 2: $n - k \neq h + 1$.

For this case, $\xi((1-k)(n-k) + s_{n-k}) \neq \xi(s_{n-k} - k(n-k) + h + 1)$. As a result, we will encounter a pole of order 1 during the contour shift at $s_{n-1} = 1 - (1-k)(n-k)$. So

$$\Omega_{k,h} = R_3 + R_4$$

where

$$R_3 = C_3 \frac{\xi \left(1 - \frac{n}{2} + k + int\right) \xi \left(1 - \frac{n}{2} + k - int\right)}{\left|\xi \left(\frac{n}{2} + int\right)\right|^2}$$

and

$$\begin{aligned} R_4 &= \frac{2}{n^{n-2} \left|\xi \left(\frac{n}{2} + int\right)\right|^2} \frac{1}{2\pi i} \int_{(1-(1-k)(n-k)-\Delta)} f(s_{n-1}) \frac{\xi((1-k)(n-k) + s_{n-k}) \xi(s_{n-k} - k(n-k) + h + 1)}{\xi(2(1-k)(n-k) + 2s_{n-k} - n + 2k)} \\ &\quad \times \xi \left((1-k)(n-k) + s_{n-k} - \frac{n}{2} + k - int \right) \xi \left((1-k)(n-k) + s_{n-k} - \frac{n}{2} + k + int \right) ds_{n-1}. \end{aligned}$$

By Stirling's formula, one can show that

$$R_3 + R_4 = O(1).$$

□

REFERENCES

- [Ar] J. Arthur, *A trace formula for reductive groups. II. Applications of a truncation operator*, Compos. Math. **40** (1980), 87-121
- [B] D. Bump, *Automorphic Forms on $GL(3, \mathbb{R})$* , Lecture Notes in Mathematics **1083**, Springer-Verlag (1984)
- [Bu] L.A. Bunimovich, *On the Ergodic Properties of Nowhere Dispersing Billiards*, Commun Math Phys. **65** (1979) 295-312.
- [BS] L.A. Bunimovich, Ya. G. Sinai *Markov Partitions for Dispersed Billiards*. Commun Math Phys. **78** (1980) 247-280.
- [Co] Y. Colin de Verdière, *Ergodicité et fonctions propres du laplacien*, Com. Math. Phys., **102**(1985), 497-502.
- [Go] D. Goldfeld, *Automorphic forms and L-functions for the group $GL(n, \mathbb{R})$* , Cambridge studies in advanced mathematics **99** (2006)
- [GR] I. S. Gradshteyn, I. M. Ryzhik, *Table of Integrals, Series, and Products*, seventh edition, Academic Press, Elsevier (2007)
- [HR] D.A. Hejhal, D. Rackner, *On the topography of Maass wave forms*, Exper. Math. **1**(1992), 275-305.
- [IT] K. Imai and A. Terras, *The Fourier Expansions of Eisenstein Series for $GL(3, \mathbb{Z})$* , Trans. AMS **273** (1982), #2, 679-694
- [I] H. Iwaniec, *Spectral Methods of Automorphic Forms*, Graduate Studies in Mathematics **53**, American Mathematical Society (2002)
- [L] R. P. Langlands, *On the functional equations satisfied by Eisenstein series*, Lecture Notes in Mathematics, **544**, Springer-Verlag, Berlin-New York, (1976)
- [Li] E. Lindenstrauss, *Invariant measures and arithmetic quantum unique ergodicity*, Ann. of Math. (2) **163** (2006), 165-219.
- [LS] W. Luo, P. Sarnak, *Quantum Ergodicity of Eigenfunctions on $PSL_2(\mathbb{Z}) \backslash \mathbb{H}^2$* , Inst. Hautes Études Sci. Publ. Math. **81** (1995), 207-237
- [MW] C. Moeglin, J.-L. Waldspurger, *Spectral decomposition and Eisenstein series*, Cambridge University Press, **113** (1995)
- [PK] R.B. Paris, D. Kaminski, *Asymptotics and Mellin-Barnes Integrals*, Encyclopedia of Mathematics and Its Applications, **85** (2001)
- [Ra] S. Ramanujan, *Some formulae in the arithmetic theory of numbers*, Messenger of Math., **45** (1916), 81-84.
- [RS] Z. Rudnick, P. Sarnak, *The behavior of eigenstates of arithmetic hyperbolic manifolds*, Com. Math. Phys., **161** (1994), 195-213.
- [S] K. Soundararajan, *Quantum unique ergodicity for $SL_2(\mathbb{Z}) \backslash \mathbb{H}$* . Ann. of Math. (2) **172** (2010), no. 2, 1529-1538.
- [Sh] A. I. Schnirelman, *Ergodic properties of eigenfunctions*, Usp. Math. Nauk., **29**(1974), 79-98
- [SV1] L. Silberman, A. Venkatesh, *On quantum unique ergodicity for locally symmetric spaces*. Geom. Funct. Anal. **17** (2007), no. 3, 960-998.
- [SV2] L. Silberman, A. Venkatesh, *Entropy bounds and quantum unique ergodicity for Hecke eigenfunctions on division algebras*. <https://arxiv.org/abs/1606.02267>
- [Ve] A. Venkov, *The Selberg trace formula for $SL(3, \mathbb{Z})$* Dokl. Akad. Nauk SSSR **228** (1976), No. 2, 273-276
- [Ze1] S. Zelditch, *Uniform distribution of eigenfunctions on compact hyperbolic surfaces*, Duke Math. Jnl., **55**(1987), 919-941
- [Ze2] S. Zelditch, *Selberg trace formulae and equidistribution theorems*, Memoirs of AMS, Vol.96, No.465, 1992.

10 HILLHOUSE AVE. NEW HAVEN, CT 06511

E-mail address: liyang.zhang@yale.edu